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ANALYTICAL AND FINITE ELEMENT MODELLING OF BUCKLING AND POST-BUCKLING OF DELAMINATED ORTHOTROPIC PLATES

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ABSTRACT

Delamination is one of the most common failure modes of laminated composite materials, and can be caused by manufacturing defects or impact loading. Under compression, a delaminated composite plate may buckle and tend to enlarge the delaminated area, which can lead to loss of global structural stability. This paper presents the elastic buckling and post-buckling analysis of an axially loaded beam-plate with an across-the-width delamination, located at a given depth below the upper surface of the plate. The analysis is done by considering a layered orthotropic plate containing through-width delamination, subjected to in-plane compression. The problem is analyzed by two approaches: (i) Through an analytical model using Fourier integral transform for determining the strain energy release rate and (ii) Through a finite element representation of the problem using MSC/ NASTRAN and where the buckling load is computed.

1. Introduction

The delamination phenomenon of composite materials is one of the most important modes of failure in composite structures, and hence it plays a crucial role in assessing the compressive response of such structures. The elastic buckling and post-buckling analysis of an axially loaded beam-plate having an across-the-width delamination is studied in Refs. [1,2,3]. Delamination tolerance in composite panel using modified virtual crack closure integral to estimate Strain Energy Release Rate (SERR) components at the delamination front from finite element output is presented in Ref.[4]. In the present work, an analytical technique is developed for determining the SERR after local buckling for a layered orthotropic beam-plate with single delamination, as shown in Figure1. Figure 2 shows the longitudinal section of the plate in the xz-plane showing delamination location across the depth. For various delamination lengths and axial loads, the buckling load is obtained using analytical approach and compared with those of finite element approach.

2. Buckling load of a delaminated clamped plate with single delamination and uniformly compressed in the axial direction

A layered orthotropic elastic beam-plate of thickness *T*, width b and length *a*, with a single delamination at depth *t (t< T/2)* from the top surface of the plate is considered. The plate is assumed to be clamped on both the edges and subjected to an axial compressive force P at the ends. The delamination is symmetrically located w.r.t. the z-axis of the beam-plate near top surface as shown in Figures1and 2. The delamination is rectangular and extends across the entire plate width. Linear stability equations of equilibrium for layered orthotropic plate are given by:

$$\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0\,;\ \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0\,;\quad D\nabla^4 w - (N_x w_{,xx} + 2N_{xy} w_{,xy} + N_y w_{,yy}) = 0$$

N_x, N_y and N_{xy} are normal and shear forces per unit width, w is the transverse displacement and D is the flexural rigidity of the plate given by $D = \dfrac{E_{11} T^3}{12(1-\upsilon_{13}\upsilon_{31})}$

The plate is subjected to an in-plane compressive load P uniformly distributed along the edges x= -a/2 and x = +a/2, and the normal and shear forces per unit width are given by $N_x = P/b$; $N_{xy} = N_y = 0$

The boundary conditions along the edges x= -a/2 and x = +a/2 are given by $w = w_{,x} = 0$.

Euler buckling load in the delamination region *c* at depth *t* is given by [5],

$$P' = \frac{\pi^2 E_{11} bt^3}{3(1-\nu_{13}\nu_{31})c^2}$$

Global buckling load at the time of local buckling is given by [5]:

$$P = \frac{\pi^2 E_{11} bTt^2}{3(1-\nu_{13}\upsilon_{31})c^2} = P^*; \quad \text{When } P = \frac{T}{t}P' \qquad \text{--- (1)}$$

Prior to local buckling the deflection of the plate is given by $u = \dfrac{Pa}{E_{11}bT}$

At the point of local buckling the deflection is given by $u^* = \dfrac{P^* a}{E_{11}bT}$

After local buckling the deflection is given by:

$$u_p = u^* + \frac{(P-P^*)c}{E_{11}b(T-t)} + \frac{(P-P^*)(a-c)}{E_{11}BT}, P > P^*$$

$$\Rightarrow u_p = P\left[\frac{ct + a(T-t)}{E_{11}b(T-t)T}\right] - \frac{\pi^2 t^3}{3c(T-t)(1-\upsilon_{13}\upsilon_{31})}, \quad P > P^* \qquad \text{--- (2)}$$

3. Formulation of the delamination problem after local buckling

Let us consider the bending of an elastic plate due to local buckling at the delamination segment R, in the xz-plane of the plate. Let the co-ordinate system be so chosen that the x-axis coincides with the line on which the delamination is located in the xz-plane. From axial displacement given by equation (2), the stress components σ_{zz} and σ_{xz} are calculated. These stress components along the delamination segment, R, are defined by the functions $G^*(x)$ and $H^*(x)$ as given by the following equations:

$$\sigma_{xz}(x,0) = \frac{G^*(x)}{2}, x \in R \quad ; \sigma_{zz}(x,0) = \frac{H^*(x)}{2}, x \in R$$

We define the displacement boundary conditions in terms of the derivative of the displacements *u, w* outside the delamination segment *S* as given by:

$$A(x) = 0, x \in S \qquad B(x) = 0, x \in S$$

$$A(x) = \frac{\partial}{\partial x}\left[u^{(1)}(x,0) - u^{(2)}(x,0)\right], \ x \in R; B(x) = \frac{\partial}{\partial x}\left[w^{(1)}(x,0) - w^{(2)}(x,0)\right], \ x \in R$$

The superscripts (1) and (2) denote the components in the upper and lower delamination segments respectively.

4. Solution of the delamination problem

Using Fourier transform we solve the equations of equilibrium

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} = 0; \ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0$$

If we define the stress components in xz- plane in terms of the Airy stress function φ as given by

$$\sigma_{xx} = \frac{\partial^2 \varphi}{\partial z^2}; \ \sigma_{zz} = \frac{\partial^2 \varphi}{\partial x^2}; \ \sigma_{xz} = -\frac{\partial^2 \varphi}{\partial x \partial z}$$

The compatibility condition reduces to

$$\nabla^4\varphi = 0\,;\ \nabla^4\varphi = \frac{\partial^4\varphi}{\partial x^4} + 2\Delta_1\frac{\partial^4\varphi}{\partial x^2\partial z^2} + \Delta_2\frac{\partial^4\varphi}{\partial z^4}\ ;\ \Delta_1 = \frac{C_{11}C_{22} - C_{12}{}^2 - 2C_{12}C_{66}}{2C_{22}C_{66}};\Delta_2 = \frac{C_{11}}{C_{22}}$$

C_{11}, C_{12}, C_{22} and C_{66} are material stiffness coefficients. Taking Fourier transformation of the bi-harmonic equation we get the ordinary differential equation in $G(\xi, z)$ as given by

$$\frac{d^4G(\xi,z)}{dz^4} - 2\Delta_1\xi^2\frac{d^2G(\xi,z)}{dz^2} + \Delta_2\xi^4G(\xi,z) = 0$$

If $G^{(1)}$ and $G^{(2)}$ are the solutions of the above equation for $z > 0$ and $z < 0$, then these functions are given by, $G^{(1)}(\xi,z) = P_1(\xi)e^{-t_1|\xi|z} + zQ_1(\xi)e^{-t_2|\xi|z} \qquad z > 0$

$$G^{(2)}(\xi,z) = P_2(\xi)e^{-t_1|\xi|z} + zQ_2(\xi)e^{-t_2|\xi|z} \qquad z < 0$$

$G^{(1)}$ and $G^{(2)}$ are the Fourier transformation of $\varphi(x,z)$ for $z > 0$ and $z < 0$. $P_1(\xi), P_2(\xi), Q_1(\xi), Q_2(\xi)$ are the unknown functions to be determined. The stress components in terms of $G^{(1)}$ are:

$$\sigma_{zz}^{(1)}(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\xi^2G^{(1)}(\xi,z)e^{-i\xi x}d\xi,\ \ z > 0\,;\ \sigma_{xz}^{(1)}(x,z) = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\xi\frac{\partial G^{(1)}(\xi,z)}{\partial z}e^{-i\xi x}d\xi,\ z > 0$$

Similarly we get the stress components in the delamination lower segment $z < 0$. The stress boundary conditions outside the delamination segment are given by,

$$\sigma_{zz}^{(1)}(x,0) - \sigma_{zz}^{(2)}(x,0) = 0; \qquad \sigma_{xz}^{(1)}(x,0) - \sigma_{xz}^{(2)}(x,0) = 0,$$

Taking Fourier transforms of the stress boundary conditions and displacement boundary conditions we solve the unknowns $P_1(\xi), P_2(\xi), Q_1(\xi), Q_2(\xi)$ in terms of $\overline{A}(\xi)$ and $\overline{B}(\xi)$, the Fourier Transforms of *A(x)* and *B(x)*.

Substituting the values of $P_1(\xi), P_2(\xi), Q_1(\xi), Q_2(\xi)$ into $G^{(1)}$ and $G^{(2)}$ we get transverse normal stress component in terms of $\overline{A}(\xi)$ and $\overline{B}(\xi)$

$$\sigma_{zz}(x,z) = \frac{\Delta_0}{4\pi(t_2{}^2 - t_1{}^2)}\int_{-\infty}^{\infty}\left\{\left[\overline{A}(\xi)y|\xi| - \frac{i\overline{B}\,\mathrm{sgn}(\xi)}{t_1}\right]e^{-t_1|\xi|z} - \left[\overline{A}(\xi) - \frac{i\overline{B}\,\mathrm{sgn}(\xi)}{t_2}\right]e^{-t_2|\xi|z}\right\}e^{-i\xi x}d\xi,$$

where, $\Delta_0 = C_{11} - \frac{C_{12}^2}{C_{22}}$; Performing the inner integral in terms of *A(s)* and *B(s)*, we get the inter-laminar normal stress components as given by,

$$\sigma_{zz}(x,z) = \frac{\Delta_0}{2\pi(t_2{}^2 - t_1{}^2)}\int_{-\infty}^{\infty}A(s)z\left\{\frac{t_1}{\left[(x-s)^2 + t_1{}^2z^2\right]} - \frac{t_2}{\left[(x-s)^2 + t_2{}^2z^2\right]}\right\}ds$$

$$-\int_{-\infty}^{\infty}\frac{B(s)(x-s)}{t_1t_2}\left\{\frac{t_1}{\left[(x-s)^2 + t_1{}^2z^2\right]} - \frac{t_2}{\left[(x-s)^2 + t_2{}^2z^2\right]}\right\}ds\,,$$

The constants t_1 and t_2 are the roots with the real parts of the quartic equation $t^4 - 2\Delta_1t^2 + \Delta_2 = 0$

Similarly the inter-laminar shear stress component is given by,

$$\sigma_{xz}(x,y) = \frac{\Delta_0}{2\pi(t_2{}^2 - t_1{}^2)}\int_{-\infty}^{\infty}\left[A(s)(x-s) + B(s)z\right]\left\{\frac{t_1}{\left[(x-s)^2 + t_1{}^2z^2\right]} - \frac{t_2}{\left[(x-s)^2 + t_2{}^2z^2\right]}\right\}ds$$

The limiting values as $z \to 0+$ and $z \to 0-$ of the stress components along the delamination line are given by,

$$\sigma_{zz}(x,0) = -\frac{\Delta_0}{2\pi(t_1+t_2)}\int_{-\infty}^{\infty}\frac{B(s)}{x-s}ds \;;\quad \sigma_{xz}(x,0) = -\frac{\Delta_0}{2\pi(t_1+t_2)}\int_{-\infty}^{\infty}\frac{A(s)}{x-s}ds$$

Using the displacement boundary conditions, we get the following singular integral equations:

$$\int_{-c}^{c}\frac{A(s)}{x-s}ds = \frac{-\pi(t_1+t_2)}{\Delta_0}G^*(x),\quad |x|<c \quad \text{and} \quad \int_{-c}^{c}\frac{B(s)}{x-s}ds = \frac{-\pi t_1 t_2(t_1+t_2)}{\Delta_0}H^*(x),\ |x|<c$$

Solutions of these singular integral equations give the following stress components:

$$\sigma_{zz}(x,0) = \frac{\mathrm{sgn}(x)}{2\pi\sqrt{\left(x^2-c^2\right)}}\int_{-c}^{c}\frac{H^*(x)\sqrt{\left(c^2-t^2\right)}}{t-x}dt,\ |x|>c\ ;\quad \sigma_{xz}(x,0) = \frac{\mathrm{sgn}(x)}{2\pi\sqrt{\left(x^2-c^2\right)}_0}\int_{-c}^{c}\frac{G^*(x)\sqrt{\left(c^2-t^2\right)}}{t-x}dt \quad \text{---(3)}$$

which are the inter-laminar normal and shear stress components outside the delamination segment S of the x-axis. The axial strain component along the delamination segment after local buckling is given by

$$\varepsilon_x^0 = \left[\frac{Pt}{E_{11}b(T-t)T}\right] + \frac{\pi^2 t^3}{3c^2(T-t)(1-\upsilon_{13}\upsilon_{31})} \qquad P > P^*$$

From the above equation we get the stress components $G^*(x)$ and $H^*(x)$ along the delamination segment. Substituting the values $G^*(x)$ and $H^*(x)$ in (3) we get the stress components outside the delamination segment. The strain energy release rate at the delamination tip after local buckling is given by,

$$G = \left(\left[\frac{Pt}{E_{11}b(T-t)T}\right] + \frac{\pi^2 t^3}{3c^2(T-t)(1-\upsilon_{13}\upsilon_{31})}\right)\left(\left[\frac{Pt}{E_{11}b(T-t)T}\right] + \frac{\pi^2 t^3}{3c^2(T-t)(1-\upsilon_{13}\upsilon_{31})}\right)\left[\left((G_{13})+E_{11}\right)\upsilon_{13}+E_{11}\right],\quad P>P^*$$

Prior to local buckling, the axial displacement the plate is given by, $u = \dfrac{Pa}{E_{11}bT}$ for $P < P^*$

which is independent of delamination length *c*; the energy release rate is zero prior to local buckling.

Strain energy release rate is computed for various applied loads and delamination lengths (Table-1) and compared with those of Ref [2] with good agreement.

5. Numerical Verification

In order to validate the above analytical procedure, finite element analysis is carried out using MSC/NASTRAN. The buckling load is calculated using the two-dimensional finite element plate model with delamination using CQUAD4 elements; the deformation contour after local buckling in shown in Fig.3. The normalized delamination buckling load for different delamination lengths and at different depths are computed and compared with the present analytical values as displayed in Tables (2) - (3), where P_0 is the critical buckling load: $P_0 = \dfrac{\pi^2 D}{a^2}$

Geometric parameters:

Length of the plate, a = *150 mm*
Width of the plate, *b = 50 mm*
Thickness of the plate, T = *3.0 mm*
Delamination thickness, t = *0.6 mm*

Material properties

E_{11}= 135.4Gpa, E_{22}= E_{33}= 9.6GPa
G_{12}= G_{13}= 4.8GPa G_{23}= 3.2GPa
v_{12}= v_{13}= 0.31 v_{23}= 0.52

6. Conclusions

A simple method for determining the analytical expression for stain energy release rate after local buckling in a layered orthotropic elastic beam-plate with a single delamination located near the top surface of the plate is presented. The problem is reduced to that of solving singular integral equations using integral transform approach, and SERR is derived in a closed form expression, and compared with other results from literature. The buckling load obtained from finite element approach is also compared with analytical values. The present approach gives an explicit closed form expression for strain energy release rate using integral transform technique with elasticity theory.

7. References

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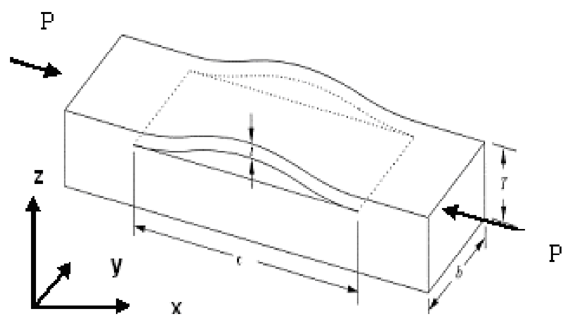

Fig.1. Geometry for orthotropic plate model with across-the-width delamination

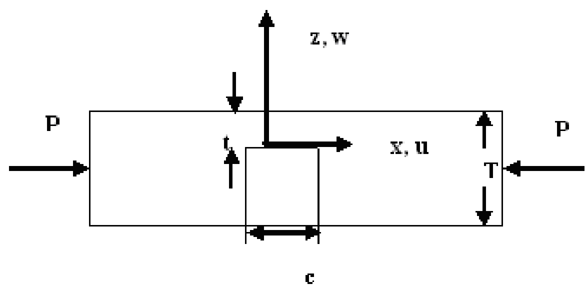

Fig.2. Section of the plate in xz-plane showing delamination

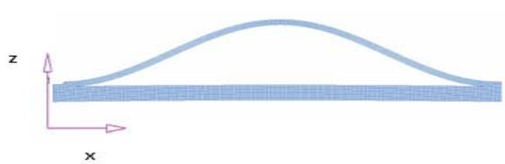

Fig.3. Post buckling behavior of delaminated plate

Table1: Comparison of Energy release rate at local buckling (a= plate length)

t/T=0.2	c/a=0.35	c/a=0.4	c/a=0.5
Present analytical	1.24	1. 05	0.854
Ref[2]	1.272	1.272	1.108

t/T	c/a	FEM	Present analytical
0.25	0	1	1
0.25	0.2	0.9809	0.99
0.25	0.4	0.3725	0.37
0.25	0.6	0.1786	0.17
0.25	0.8	0.1074	0.1

Table 2: Normalized buckling loads (P/P_0) for different delamination lengths

c/a	t/T	FEM	Present analytical
0.5	0.2	0.1666	0.1585
0.5	0.3	0.3428	0.3469
0.5	0.4	0.5497	0.5675
0.5	0.5	0.6855	0.6895

Table 3: Normalized buckling loads (P/P_0) for delamination at different Depths