A C₀ CONTINUOUS FOUR-NODED CYLINDRICAL SHELL ELEMENT

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Abstract—A C₀ continuous four-noded cylindrical shell element with independent interpolations for in-plane displacements u and v; transverse displacement w and face rotations θ₁ and θ₂ are made efficient by using substitute smoothed shape functions for w or membrane strain evaluation. This removes “membrane locking”, making it the simplest efficient quadrilateral cylindrical shell element available.

1. INTRODUCTION

A series of very interesting studies have appeared recently [1–4], which show that it is possible to improve dramatically, the behaviour of doubly curved thin shell elements by using a “reduced interaction” method in which the membrane strains are evaluated using a reduced interpolation for the transverse displacement w. In this paper we demonstrate that for a cylindrical shell element, this can be achieved by using a smoothed strain interpolation for the membrane strains. The dramatic improvement in accuracy is due to the removal of membrane locking, a phenomenon which has been recently studied and established for thin curved beam and arch elements [5–7].

The difficulty of deriving simple doubly curved or cylindrical deep shell elements, based on independently interpolated low order displacement elements, has been excellently reviewed in Ref. [8]. Successful elements required either high-order interpolations [9–11] or had to be based on independent-strain models and hence coupled displacement fields [12]. The dramatic success achieved with the use of a substitute reduced interpolation for the transverse displacement w in the membrane strain energy evaluation of a doubly curved shell element [1] seems to indicate that this is due to the removal, thereby, of membrane locking; a phenomenon that was shown to exist for thin curved beam elements [5–7]. This suggests that a very simple four-noded rectangular cylindrical shell element of 20 dof, using independent bilinear interpolations for the five external nodal displacements u, v, w, θ₁, and θ₂, can be made competitive with the well-known coupled-displacement field independent strain 20 dof model of Ashwell and Sabir [12] and even with the higher order 48 dof model of Bogner, Fox, and Schmid [13]. This is made possible by the use of substitute smoothed interpolation functions in the membrane and shear strain evaluations so that both membrane and shear locking are removed. The pinched cylinder problem is used as a test example and the nature of membrane locking is established.

2. ELEMENT DESCRIPTION

The cylindrical shell element presented here is a conforming (i.e. C₀ continuous) shear-flexible rectangular version of a nonconforming (in relation to the C₀ continuity required) cylindrical thin-shell element presented in Ref. [12]. The five external nodal displacements chosen at each node are u, v, w, θ₁, and θ₂ (see Fig. 1). The membrane strains, curvatures, and shear strains are then

\[ \epsilon_u = \epsilon_{u1} + w/R, \]
\[ \epsilon_v = \gamma_v, \]
\[ \epsilon_w = w, \]
\[ \chi_u = 0, \]
\[ \chi_v = 0, \]
\[ \chi_w = 0, \]
\[ \gamma_{uv} = 0, \]
\[ \gamma_{vw} = 0, \]
\[ \gamma_{wu} = 0, \]
\[ \gamma_{uv} = \gamma_{vw} = \gamma_{wu} = 0. \] (1)

So that in the thin shell limit where \( \gamma_{uv} = \gamma_{vw} = \gamma_{wu} = 0 \), we recover the strain-displacement equations given in Timoshenko and Woinowsky-Krieger [14].

The strain energy density is now

\[ W = \frac{Et}{2(1 - v)} \left\{ \epsilon_u^2 + \epsilon_v^2 + 2(1 - v) \left( \epsilon_u \epsilon_v + \frac{\gamma_{uv}^2}{2} \right) \right\} + \frac{1}{12}(\chi_u^2 + \chi_v^2) \left\{ 2(1 - v)\chi_u \chi_v - \frac{\gamma_{uv}^2}{4} \right\} + \frac{4G\tau}{2}(\gamma_{uv} + \gamma_{vw} + \gamma_{wu}). \] (2)

Figure 2 shows the isoparametric system used. In order to derive the stiffness matrix, it is necessary to interpolate these strains in terms of the displacement variables u, v, w, θ₁, and θ₂. In this paper...
3. FIELD CONSISTENCY AND SPURIOUS CONTRAINING

As the phenomenon of shear locking is quite well established, we shall consider first the effect of the introduction of these simple shape functions into one of the shear strains. We notice that for a rectangular element aligned with the global (x, y) axes, we can write

\begin{align*}
\delta_{i} &= u_{i} + a_{1}x + a_{2}y + a_{3}x y, \\
w &= u_{4} + a_{4}x + a_{5}y + a_{6}x y.
\end{align*}

so that the shear strain \( \gamma_{c} \) becomes

\begin{equation}
\gamma_{c} = (a_{9} - b_{1}) + (a_{4} - b_{3})y + a_{1}x + a_{0}x y.
\end{equation}

We notice now that the coefficients associated with the constant and linear in y terms comprise quantities from both interpolation functions. However, the coefficients associated with the linear in x and with the product \( x y \) terms originate only from the interpolation for the face rotation \( \theta_{b} \). If an exact integration of the shear strain energy had been performed, then one can expect that in the penalty function limits that are reached as the thickness of the plate or shell is made very small, the following four constraints are in some manner enforced:

\begin{align}
a_{0} - b_{1} &= 0, \\
0 &= b_{3} - a_{4}, \\
a_{1} &= 0,
\end{align}

Clearly, the last two constraints will now introduce a spurious in-plane stiffening state of stress that was described as 'in-plane' or membrane locking and this was clearly demonstrated for the curved beam and arch element [5, 6]. In the next section, we shall establish for a simple shallow shell element that by using smoothed interpolation functions for \( w \) in the membrane strains, and for \( \theta_{b} \) in the shear strains, we can arrive at an element that is free of both membrane and shear locking.

4. SMOOTHED INTERPOLATION FUNCTIONS

In Refs. [5–7], we have seen that the spurious constraints can be eliminated by an optimal integration strategy that will use separate numerical integration rules for the different strain energy components. Thus, the spurious constraints originate from the membrane strain energy. These terms are integrated by an order low enough to remove the inconsistent terms but accurate enough to retain all consistent terms. The bending energy is, of course, evaluated exactly.

Alternatively, if a properly chosen substitute shape function procedure is used, then a single and exact integration strategy can be used uniformly for all energy terms, and this will ensure the presence of all true constraints. We shall adopt this strategy here. From Eqns. (13), we notice that the strains that must be consistently interpolated are \( \gamma_{c} \) and \( \gamma_{y} \), so that need a substitute function for \( w \) consistent with \( \gamma_{c} \), a substitute function for \( \theta_{b} \), and \( \gamma_{y} \), and a substitute function for \( \theta_{b} \), consistent with \( \gamma_{y} \). We require, therefore, two substitute functions of the form

\begin{align}
h_{1} &= h_{0} + h_{1}, \\
h_{2} &= h_{0} + h_{2}.
\end{align}

These interpolations form the basis of the reduced interpolation technique adopted here so that both membrane and shear locking are avoided.

5. NUMERICAL EXPERIMENTS

The element stiffness matrix is now computed in a straightforward manner, using uniformly, a 2 × 2 Gaussian integration scheme. Options are introduced, which allow various reduced interpolation schemes to be investigated. Of particular interest in the present study are the following three schemes:

Scheme A—element proposed in this paper, without any locking by using reduced interpolations for membrane and shear strains.

Scheme B—smoothed interpolation of shear strains alone is introduced, thus producing an element in which membrane locking is present.

Scheme C—original shape functions are used throughout, and so the element has both membrane and shear locking.

The test problem chosen is the pinched cylinder shown in Fig. 3. This has been used most often as
the standard for evaluating cylindrical shell elements. One octant is considered and the geometrical and elastic properties considered are: radius is \( r = 4.953 \text{ in.} \), thickness \( t = 0.094 \text{ in.} \), length is \( 10.5 \text{ in.} \), \( E = 5 \times 10^6 \text{ psi} \), \( v = 0.3212 \), and a load of \( P = 100 \text{ lb} \) is applied in both the thick and thin cylinders. The exact solution \[14\] is based on an inextensional theory and gives the deflections under this load as \( w_1 = 0.1084 \text{ in.} \) and \( w_2 = 0.2428 \times 10^{-6} \text{ in.} \).

Table 1 shows the results obtained as the octant is idealised by \( I \times N \) elements, where \( N \) is the number of divisions along the curved edge of the octant and only one division is used along the length. This offers a ready comparison with identical idealisations used for the 48 dof element of Bogner et al. \[13\]. We notice that the reduced interpolated element scheme A is as efficient as the high precision element on a degree-of-freedom basis. Also noteworthy, is the very poor behaviour shown by the original element scheme C—the locking makes the results virtually meaningless. Table 2 compares the locking effect in the thick \((t = 0.094 \text{ in.})\) and thin \((t = 0.01548 \text{ in.})\) cases, respectively. The additional stiffening parameter due to locking is defined as

\[ e = \frac{w_{\text{thick}}}{w_{\text{thin}}} - 1 \]

and the very large values of \( e \) reflect errors approaching 100%. The remarkable agreement between the ratio of additional stiffening parameters for the thick and thin cases \((e_{\text{thick}})\) and \((e_{\text{thin}})\)

Table 2. Nature of shear + membrane locking in a thick and a thin pinched cylinder

\[ \begin{array}{cccccc}
N & w_1 & w_2 & t_1 & t_2 & e_{\text{thick}} \\
1 & 0.9664 \times 10^{-4} & 0.5112 \times 10^{-4} & 31.21 & 1121.5770 & 36.9 \\
2 & 2.531 \times 10^{-4} & 3.172 \times 10^{-4} & 312.13 & 1893 & 36.9 \\
4 & 0.127 \times 10^{-4} & 0.127 \times 10^{-4} & 627.289 & 7389 & 37.0 \\
8 & 0.064 \times 10^{-4} & 0.064 \times 10^{-4} & 76.862 & 2513 & 37.0 \\
16 & 0.3428 \times 10^{-4} & 0.3428 \times 10^{-4} & - & - & 36.9 \\
\end{array} \]

\[ \text{Reference [14]} \]

shows that the errors introduced when scheme C is used are essentially those due to locking, a typical feature of which is that they are exaggerated in a 10^{th} fashion, as seen here.

We next examine the element for the membrane locking behaviour. This can be demonstrated by comparing the results obtained from schemes A, B, and C, as the thickness \( t \) of the shell is varied from a reasonably thick to a reasonably thin value. Figure 4 plots the logarithmic scale data, the theoretically predicted deflection under the load, and the finite element results \( w_A, w_B, \) and \( w_C \) from a 1 \times 8 mesh. It is seen that the \( w_C \) line is very close to the exact values, and that it follows very accurately a \( t^{0.5} \) relationship that is to be expected when the bending is of an inextensional type. We also find that \( w_A = w_B \) follow a \( t^{-1/4} \) rule, showing that an additional locking stiffness of \( t \) nature is introduced in the membrane and shear locking mechanisms. Clearly, the original element without reduced interpolation or even with only shear locking removed, will be too poor in performance to be used in any practical case.

4. CONCLUSIONS

We have demonstrated here, how a simple \( C_0 \) continuous rectangular cylindrical shell element, based on the lower order bi-linear polynomial interpolation, without the trigonometric rigid body terms, can be made useful by a reduced interpolation technique for the membrane and shear strains. This also explains the remarkable improvement noticed in the "reduced interaction" technique of Mohr \[1–4\] where the use of a similar substitute lower order function for \( w \) in the membrane strains results in an improved element. It is quite simple to extend these concepts to derive a doubly curved shell element using simple polynomial interpolations and with optimal reduced interpolation so that there is no membrane or shear locking.

**REFERENCES**