Engineering Notes

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Some Remarks on the Solution of the Lifting Line Equation

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Introduction

SO much has been said about Prandtl's lifting line equation that it seems futile to elaborate on it further. However, the author feels that a few salient features of this equation have not been remarked upon earlier. The equation is[1]

$$\Gamma(y) = \pi U C(y)[\alpha(y) - \epsilon(y)] \tag{1}$$

where

$$\epsilon(y) = \frac{1}{4\pi U}\int_{-s}^{s} \frac{d\Gamma}{d\eta} \times \frac{d\eta}{(y-\eta)} \tag{2}$$

and

$$\Gamma(s) = \Gamma(-s) = 0 \tag{3}$$

Here Γ is the local circulation, α the local geometric angle of attack measured from the zero lift line, C the local chord, ϵ the local downwash induced by the trailing vortices, U the freestream velocity, s the wing semispan, and y the spanwise coordinate.

Equation (1) is usually solved using a sine series for Γ in the form

$$\Gamma(\theta) = \sum_{n=1}^{\infty} A_n \Gamma_n(\theta) = 2sU \sum_{n=1}^{\infty} A_n \sin n\theta \tag{4}$$

in a collocation method, and $\theta = \cos^{-1}(y/s)$.

Since Γ and α are related by a linear operator, it may be shown that

$$\alpha(\theta) = \sum_{n=1}^{\infty} A_n \alpha_n(\theta) = \frac{2s}{\pi C(y)} \sum_{n=1}^{\infty} A_n \sin n\theta + \epsilon(\theta) \tag{5}$$

and

$$\epsilon(\theta) = \sum_{n=1}^{\infty} A_n \epsilon_n(\theta) = \frac{1}{2} \sum_{n=1}^{\infty} A_n n \sin n\theta / \sin\theta \tag{6}$$

The lift and drag coefficients are, respectively,

$$C_L = \pi A R A_1 / 2 \tag{7}$$

$$C_D = \pi A R \sum_{n=1}^{\infty} n A^2{}_n / 4 \tag{8}$$

where the aspect ratio $AR = 4s^2/S$, and S is the wing area.

We note that C_D contains only the square of the unknown constants A_n. In other words, the loadings (Γ_n, α_n) are orthogonal in the sense used by Graham.[3] Orthogonality is usually associated with simplicity, and this is reflected in Eqs. (7) and (8).

Let us assume that a loading (Γ, α) is expressible by a finite number of terms m of Eqs. (4) and (5) to the desired accuracy. Then from the orthogonality of the loadings it may be easily shown that

$$\begin{aligned}\int_{-s}^{s} \Gamma(y)\epsilon_n(y)dy &= \int_{-s}^{s} \epsilon(y)\Gamma_n(y)dy \\ &= A_n \int_{-s}^{s} \Gamma_n(y)\epsilon_n(y)dy \\ &= A_n \frac{\pi}{2} s^2 U n\end{aligned} \tag{9}$$

for $n = 1, \ldots, m$.

Hence if the circulation distribution is given, the unknown constants may be obtained from

$$A_n = 2\int_{-s}^{s} \Gamma(y)\epsilon_n(y)dy / (\pi s^2 U n) \tag{10}$$

to yield $\alpha(y)$ from Eqs. (5) and (6). If the downwash distribution $\epsilon(y)$ is given, the constants may be obtained from

$$A_n = 2\int_{-s}^{s} \epsilon(y)\Gamma_n(y)dy / (\pi s^2 U n) \tag{11}$$

to yield $\Gamma(y)$ from Eq. (4). However, more frequently the wing geometry is given in which case $\alpha(y)$ is specified. Using Eq. (5) to substitute for $\epsilon(y)$ in Eq. (11) results in m linear simultaneous equations

$$\begin{aligned}A_n \frac{\pi}{2} s^2 U n + (1/\pi U)\int_{-s}^{s} \Big[\Gamma_n(y) \sum_{r=1}^{m} A_r \Gamma_r(y)/C(y)\Big] dy \\ = \int_{-s}^{s} \alpha(y)\Gamma_n(y)dy\end{aligned} \tag{12}$$

which may be solved for the A_n. Results of Eqs. (10) and (11) are important since they allow the A_n to be evaluated explicitly. Equation (12) clearly shows that for elliptic planforms, i.e. $C(\theta) \sim \sin\theta$, the A_n can be obtained explicitly for any angle of attack distribution. In this form it is a generalization of Filotas' results.[2] It is further interesting to note that Eq. (12) is the same as would be obtained by an application of the Galerkin method[4] to Eq. (1). The present derivation shows that the Galerkin approach is a very natural one and is the reason for its success in Ref. 4.

The advantage of using Eqs. (10–12) is that the solution does not depend only on the characteristics of the wing at a small number of isolated points as in a collocation procedure, but gives an approximate solution along the entire span. Hence discontinuities, flap deflections, etc., may be accounted for.

An Example—Rectangular Wing

Application of Eq. (12) to a wing of constant chord C, and constant incidence α, gives

$$\begin{aligned}\frac{\pi}{2} n A_n - \frac{16 s}{\pi C} \sum_{r=1}^{m}{}' \frac{n r A_r}{(r^2 - n^2)^2 - 2(r^2 + n^2) + 1} \\ = \pi\alpha ; \quad n = 1 \\ = 0 ; \quad n \neq 1\end{aligned} \tag{13}$$

for $n = 1, \ldots, m$;

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where Σ' denotes summation over only those terms for which $(n + r)$ is even. A one term approximation for Γ yields

$$A_1 = \pi\alpha/(\pi/2 + 16s/3\pi C) \tag{14}$$

A two term approximation shows

$$\begin{aligned} A_1 &= \pi\alpha\Big/\left[\left(\pi/2 + \frac{16s}{3\pi C}\right) - \left(\frac{16s}{15\pi C}\right)^2\Big/\left(\frac{\pi}{2} - \frac{144s}{35\pi C}\right)\right] \\ A_2 &= 0 \\ A_3 &= \frac{16s\alpha}{15C}\Big/\left[\left(\frac{\pi}{2} + \frac{16s}{3\pi C}\right)\left(\frac{\pi}{2} - \frac{144s}{35\pi C}\right) - \left(\frac{16s}{15\pi C}\right)^2\right] \end{aligned} \tag{15}$$

The computation effort required in solving Eq. (13) is less compared to a collocation method where trigonometric functions must be evaluated.

Conclusions

The method outlined in this note is valid for any set of loadings (Γ_n, α_n) which are orthogonal in Graham's sense. It may be used for non-orthogonal loadings if they are first converted to an orthogonal set as suggested by Graham.[3]

References

[1]Glauert, H., "The Elements of Aerofoil and Airscrew Theory," Cambridge University Press, New York, 1926.

[2]Filotas, L. T., "Solution of the Lifting Line Equation for Twisted Elliptic Wings," *Journal of Aircraft*, Vol. 8, No. 10, Oct. 1971, pp. 835–836.

[3]Graham, E. W., "A Drag Reduction Method for Wings of Fixed Planform," *Journal of the Aeronautical Sciences*, Vol. 19, No. 12, 1952, pp. 823–825.

[4]Anderson, R. C. and Millsaps, K., "Application of the Galerkin Method to the Prandtl Lifting Line Equation," *Journal of Aircraft*, Vol. 1, No. 3, May–June 1964, pp. 126–128.