Analysis of stabilised output error methods

G.Gopalratnam
J.R.Raol

Indexing terms: Output-error methods, Aircraft stability, Simulation

Abstract: In the paper, analysis of stabilised output-error methods (SOEMs) for parameter estimation of unstable aircraft is presented. These methods overcome the numerical difficulties encountered in parameter estimation of unstable systems by utilising measured states. The methods, along with the output error method **(OEM)** and the equation error method **(EEM),** are briefly described for the sake of comparison. However, the main idea of the paper is to present asymptotic analysis of the **SOEM.** The results of application of **SOEMs** to simulated data of an unstableiaugmented aircraft are presented.

1 Introduction

The output-error method **(OEM)** [I] is the most widely used technique for estimation of parameters of stable dynamical systems **[2,** 3]. **It** has been very successfully utilised for the estimation of stability and control derivatives of aircraft from flight data [4–6]. However, the method poses severe difficulties when applied to inherently unstable systems [7, 8]. Even if the basic unstable plant is operating with a stabilising feedback loop around it, the application of the **OEM** to directly estimate the parameters of the state-space models of the plant from its input/output data poses similar difficulties. When the system is unstable, numerical integration leads to diverging solutions.

Modem aircraft are designed to be aerodynamically unstable to meet improved performance characteristics such as high manoeuvrability. Such basically unstable aircraft are flown with a fly-by wire control system **(FBWCS),** i.e. in a closed loop. However, many applications in flight mechanics require the determination **of** the aerodynamic derivatives of the basic unstable aircraft. The aerodynamic derivatives are required **[7]** both to explain aerodynamic, stability and control behaviour of the aircraft, thereby describing its static/dynamic behaviour, and, **in** the mathematical models, for design of flight control systems and **high** fidelity simulators.

Hence, for the successful application of the **OEM** to an inherently unstable/fly-by wire control system, special techniques and modifications are necessary to prevent or arrest the growth **of** divergence. **An** approach which provides artificial stabilisation in the mathematical model used in the **OEM** (software) has **been** reported [8] for unstable aircraft. However, this approach requires an extensive engineering effort. **A** method based on an extension of the basic principles of regression analysis, called the equation-decouplimg **OEM(EDOEM)** has been applied for parameter estimation of unstable systems [9]. This method uses measured states to decouple the state equations and integrate the system of differential equations independent of each other. The decoupling of ***the*** equations may change an unstable system to a stable one. The degree of decoupling can be changed depending on the instabilities in the system. Several approaches to parameter estimation of inherently unstable/FBW aircraft have been reported [10–13].

The detailed analysis of stabilised **OEMs (SOEMs),** the **OEMs** which use measurements **of** required states to stabilise the estimation algorithm, is limited in the literature. In the present paper, such methods are termed stabilised output-error methods **(SOEMs).** After providing a brief overview **of** the **OEM** and the equation error method **(EEM),** the **SOEMs** consisting of the **RAOEM** (regression analysis **OEM)** and the **EDOEM** are described. Subsequently, detailed analysis of the **SOEM** is presented. The main idea of the paper is to give an asymptotic analysis of the **SOEMs.** The results of application of SOEMs to simulated data of an unstableiaugmented aircraft are also presented.

2 Output error method

The **OEM** minimises the error between the measured and model responses produced by identical inputs. It is assumed that there is no process noise. The **OEM** is applicable to both linear and nonlinear systems [14]. For simplicity, a linear system is described. The dynamics of the system are given as

$$X = Ax + Bu \qquad \text{with } x(0) = x_0 \quad (1)$$

$$y = Cx + Du \quad (2)$$

$$z(k) = y(k) + \nu(k) \qquad k = 1, 2, \ldots, N \quad (3)$$

where x is the n-state vector, u the control vector, y the m-observation vector, z the measurement vector, N the number of data points and v is the measurement noise assumed to be Gaussian with **zero** mean and covariance matrix **R.** The Θ $\{A, B, C, D\}$ represents the parameter vector to be estimated. To estimate the parameters Θ, the cost function to be minimised is defined as

$$E(\Theta) = \frac{1}{2}\sum_{k=1}^{N}[z(k) - y(k)]^T R^{-1}[z(k) - y(k)] + \frac{N}{2}\ln|R| \quad (4)$$

IEE Proceedings online no. 19960195

Paper first received 19th April 1995 and in revised form 21st November 1995

The authors are with the Flight Mechanics and Control Division, National Aerospace Laboratories, Bangalore – 560017, India

Here T denotes the transpose of a matrix/vector. Minimisation of the above cost llinction with respect to Θ yields the maximum likelihood estimates of Θ.

$$\hat{\Theta}_{l+1} = \hat{\Theta}_l + \mu \Delta \Theta_l \tag{5}$$

$$\Delta\Theta_l = \left[\sum_k \left(\frac{\partial y(k)}{\partial \Theta} \right)^T R^{-1} \frac{\partial y(k)}{\partial \Theta} \right]^{-1} \times \sum_k \frac{\partial y(k)}{\partial \Theta} R^{-1} (z_m(k) - y(k)) \tag{6}$$

The first term in eqn. 6 is the Gauss–Newton approximation to the second gradient of the cost function $E(\Theta)$ and is called the information matrix. Eqn. 5 in terms of the first and second gradients can be written as

$$\hat{\Theta}_{l+1} = \hat{\Theta}_l + [\nabla^2 E(\Theta)]^{-1} \nabla E(\Theta) \tag{7}$$

Here l stands for the iteration number. The constant μ is called the damping factor which can he used to improve the convergence of the algorithm. Thus, to compute the first and second gradients, we need to compute the term $\partial y(k)/\partial\Theta$. This term is called the sensitivity matrix and is obtained by the finite-difference method. For aircraft parameter estimation the OEM is the most widely used estimator. since it has many desirable statistical properties [1].

3 Equation error method (EEM)

This method is based on the principle of least squares. The EEM [11] minimises the error in the (state) equations to estimate the parameters. It is assumed that the states, their derivatives and control inputs are accurately measured. The method is fast, simple, noniterative in nature and is applicable to linear as well as linear-in-parameter (LIP) systems. The equation error can be written as (from eqn. 1)

$$e(k) = \dot{x}_m - Ax_m - Bu_m \tag{8}$$

Here x_m is the measured state, subscript m denoting 'measured'. The parameter estimates are obtained by minimising the equation error with respect to Θ. Eqn. 8 can be written as

$$e(k) = \dot{x}_m - A_a x_{am} \tag{9}$$

where A_a = **(AIR]** and $x_{am} = [x_m^T \mid u_m^T]^T$.

In this case. the cost function is given by

$$E(\Theta) = \frac{1}{2} \sum_{k=1}^{N} [\dot{x}_{am}(k) \quad\quad x_{am}(k)]^T [\dot{x}_{am}(k) - A_a x_{am}(k)] \tag{10}$$

The estimator is given as

$$\hat{\Theta}(l+1) = \hat{\Theta}(l) + \mu \sum_{k=1}^{N} (\dot{x}_{am}(k) - A_a x_{am}(k)) (x_{am}(k))^T \tag{11}$$

ignoring the information matrix part for simplicity [12].

Application of the EEM to parameter estimation requires accurate measurements of states and their derivatives. The EEM can be applied to unstable systems because it does not involve any numerical integration which would otherwise cause divergence. The utilisation of measured states (and the state-derivatives) for estimation. stabilises the algorithm, and enables estimation of the parameters of the unstable system directly. This notion of using measured states is the basis of the stabilised output error methods. The SOEMs described in this paper seem to fall in between the EEM and the OEM methods for parameter estimation. and hence can he said to belong to a class of mixed EE-OEM methods for parameter estimation.

4 Stabilised output-error methods (SOEMs)

The instability caused by numerical divergence can he overcome by incorporating stabilisation into the OEM using measured states. The manner of utilising the measured states to stabilise the system equations, leads to two varieties of stabilised OEMs: the equation decoupling OEM (EDOEM), wherein the states pertaining to the off-diagonal elements are replaced by the corresponding measured states. and the regression analysis OEM (RAOEM), wherein only the states occurring with the parameters which cause numerical divergence are replaced by the state-measurements.

4.1 Equation decoupling OEM (EDOEM)

The system matrix **A** is partitioned into two sub-matrices, denoted by A_D and A_{OD}, where **A,** is the diagonal matrix containing the diagonal elements of A, and the, matrix ***A*,,** the off-diagonal elements. Augmenting the contrbl input vector u with the measured states x_m, eqn. 1 can he written as

$$\dot{x} = A_D x + [B \mid A_{OD}$$

The only integrated variables entering the differential equation are in the ***A*,** matrix, and all other variables related to A_{OD} are measured states. Thus each differential equation can be integrated independently of the others, and hence the equations are completely decoupled. This decoupling may change the original unstable system into a stable one. Thus, application of this method requires independent, noise-free measurements of all the state variables.

The cost function to be minimised is given by eqn. **4.** However, the computation of the sensitivity function involves the decoupled matrices ***A*,** and **A„** and the state measurements augmenting the control input measurements.

4.2 Regression analysis OEM (RAOEM)

In this method, which is a variation of the EDOEM, the measured states are used for the aerodynamic derivatives which cause instability in the system, and integrated states are used as the remaining states. While this approach has the advantage of stabilisiiig the system, it needs accurate measurement of states, as in the case of the EDOEM. In this case, the matrix **A** is partitioned into two parts, A_S containing that part of the A matrix which has parameters that do not contribute to divergence and A_{US} for that part which contributes to the system instability, so that the system equation has the form

$$\dot{x} = A_S x + [B \mid A_{US}] \begin{bmatrix} u_m \\ x_m \end{bmatrix} \tag{13}$$

i.e. the integrated states are used only for the stable part of the system matrix and measured states for the parameters contributing to the unstable part of the system. Eqn. 13 has a form similar to eqn. 12 for the EDOEM. The only difference between the two is that in eqn. 12. the matrix ***A*,** is diagonal, whereas in eqn. 13 A_S may not be diagonal. Thus, in the RAOEM and the EDOEM, the measured states are used. This technique tries to prevent/arrest the growth of errors due to numerical integration of the system equations.

5 Analysis of SOEMs

In this Section, the implications of the use of measured states, in terms of sensitivity matrix computation and covariance, are studied, thereby providing an analytical basis to the working of the SOEMs. It is assumed that the analysis for the OEM is valid when applied to a stable system for which the convergence of the algorithm is generally assured. Also, it is assumed that the presented analysis for the SOEM is valid for the unstable system, since the use of measured states stabilises the parameter estimation method.

For the case where there is no process noise, eqn. 1 can be discretised and expressed as

$$x(k+1) = \phi x(k) + \psi B u(k) \qquad \text{and } x(0) = x_0 \tag{14}$$

$$y(k) = Cx(k) + Du(k) \tag{15}$$

where ϕ denotes the state transition matrix and ψ its integral

$$\phi = e^{A\Delta t} = I + A\Delta t + A^2\frac{\Delta t^2}{2!} + \dots \tag{16}$$

$$\psi = \int_0^{\Delta t} e^{A\tau} d\tau \approx I\Delta t + A\frac{\Delta t^2}{2!} + A^2\frac{\Delta t^3}{3!} + \cdots \tag{17}$$

where $\Delta t = t(k+1) - t(k)$ is the sampling interval. Computation of the parameter improvement $\Delta\Theta$, eqn. 6, requires the computation of the sensitivity matrix

$$\left(\frac{\partial y}{\partial \Theta}\right)_{ij} = \frac{\partial y_i}{\partial \Theta_j} \tag{18}$$

The sensitivity matrix, eqn. 18, is obtained from the sensitivity equations, which are obtained by partial differentiation of the system equations with respect to each element of the unknown parameter vector Θ. Since the sensitivity equations have an identical matrix A as the system equations, the same transition matrix, eqn. 16, can be used to solve them. By differentiating eqns. 14 and 15 with respect to Θ, the discrete form of the sensitivity equations are obtained as [7]

$$\frac{\partial x(k+1)}{\partial \Theta} \approx \phi\frac{\partial x(k)}{\partial \Theta} + \frac{\partial \phi}{\partial \Theta}x(k) + \psi\frac{\partial B}{\partial \Theta}u(k) + \psi B\frac{\partial u(k)}{\partial \Theta} + \frac{\partial \psi}{\partial \Theta}Bu(k) \tag{19}$$

$$\frac{\partial y(k)}{\partial \Theta} = C\frac{\partial x(k)}{\partial \Theta} + \frac{\partial C}{\partial \Theta}x(k) + \frac{\partial D}{\partial \Theta}u(k) \tag{20}$$

In order to study the implication of measured states in the SOEMs, the detailed computations of the parameter update in the OEM and the SOEM, using a simple first-order example, is considered in the discussion to follow.

The state equation of the system is given as

$$\dot{p} = L_p p + L_\delta \delta \tag{21}$$

and the measurement equation is

$$p_m(k) = p(k) + \nu(k) \tag{22}$$

where p is the roll rate and δ is the aileron deflection. The L_p and L_δ are the aerodynamic derivatives to be estimated. It is to be noted here that A, the system matrix, is L_p and B, the control matrix, is L_δ, matrix $C = 1$ and $R = 1$.

The cost function for the **OEM** for this case is given by (eqn. 4)

$$E(L_p, L_\delta) = \frac{1}{2}\sum_{k=1}^{N}[p_m(k) - p(k)]^2 \tag{23}$$

where $p(k)$ is the computed response using the *a priori* values of L_p and L_δ

$$p(k+1) = \phi p(k) + \psi B\delta(k) \tag{24}$$

Using eqns. 16 and 17, and neglecting higher-order terms of Δt, which is justified since the sampling interval is small in general, the matrices ϕ and ψ are given by

$$\phi = 1 + L_p\Delta t \quad \text{and} \quad \psi = \Delta t \tag{25}$$

Substituting eqn. 25 into eqn. 24, we get

$$p(k+1) = (1 + L_p\Delta t)p(k) + \Delta t L_\delta \delta(k) \tag{26}$$

The estimates of L_p and L_δ are those which minimise eqn. 23. The sensitivity matrix is given by eqn. 19

$$\frac{\partial p(k+1)}{\partial(L_p, L_\delta)} = \phi\frac{\partial p(k)}{\partial(L_p, L_\delta)} + \frac{\partial \phi}{\partial(L_p, L_\delta)}p(k) + \psi\frac{\partial B}{\partial(L_p, L_\delta)}\delta(k) + \psi B\frac{\partial \delta(k)}{\partial(L_p, L_\delta)} + \frac{\partial \psi}{\partial(L_p, L_\delta)}B\delta(k) \tag{27}$$

In terms of partial differentiation with respect to individual aerodynamic derivatives, using information from eqn. 25, eqn. 27 becomes (after simplification)

$$\frac{\partial p(k+1)}{\partial L_p} = \frac{\partial p(k)}{\partial L_p} + L_p\Delta t\frac{\partial p(k)}{\partial L_p} + \Delta t p(k) \tag{28}$$

$$\frac{\partial p(k+1)}{\partial L_\delta} = \frac{\partial p(k)}{\partial L_\delta} + L_p\Delta t\frac{\partial p(k)}{\partial L_\delta} + \Delta t \delta(k) \tag{29}$$

Thus, in this case, the first gradient $\nabla E(\Theta)$ is given by

$$\nabla p = \left[\frac{\partial p}{\partial L_p}, \frac{\partial p}{\partial L_\delta}\right] \tag{30}$$

The parameter vector $\Theta = [L_p, L_\delta]$ and successive estimates of Θ are obtained by iterating eqn. 5. Thus, for the single state variable case, starting with the initial estimates of the parameters L_p and L_δ which will be Θ_0, using eqn. 5, Θ_1 is first obtained and then iteratively, the subsequent estimates **of** Θ are obtained by computing the first and second gradients of eqn. 24 which are given by:

$$\nabla E(\Theta) = \begin{bmatrix} \sum_{k=1}^{N}(p_m(k) - p(k))\frac{\partial p(k)}{\partial L_p} \\ \sum_{k=1}^{N}(p_m(k) - p(k))\frac{\partial p(k)}{\partial L_\delta} \end{bmatrix} \tag{31}$$

$$= \begin{bmatrix} \sum_{k=1}^{N}(p_m(k) - p(k))\left[\frac{\partial p(k-1)}{\partial L_p} + L_p\Delta t\frac{\partial p(k-1)}{\partial L_p} + \Delta t p(k-1)\right] \\ \sum_{k=1}^{N}(p_m(k) - p(k))\left[\frac{\partial p(k-1)}{\partial L_\delta} + L_p\Delta t\frac{\partial p(k-1)}{\partial L_\delta} + \Delta t \delta(k-1)\right] \end{bmatrix} \tag{32}$$

$$\nabla^2 E(\Theta) \cong \sum_{k=1}^{N} (\nabla E(\Theta))^T \nabla E(\Theta) \tag{33}$$

$$= \left[\begin{array}{c|c} \sum_{k=1}^{N} \frac{\partial p(k)^2}{\partial L_p} & \sum_{k=1}^{N} \frac{\partial p(k)}{\partial L_p}\frac{\partial p(k)}{\partial L_\delta} \\ \hline \sum_{k=1}^{N} \frac{\partial p(k)}{\partial L_p}\frac{\partial p(k)}{\partial L_\delta} & \sum_{k=1}^{N} \frac{\partial p(k)^2}{\partial L_\delta} \end{array}\right]$$

In order to analyse the concept of SOEMs, assuming that the derivative L_p causes instability in eqn. 21, the measured states are used for the state p in addition to the measured control surface deflection δ. The expressions for the first and second gradients are now derived for this case, and the effect of the measured states on the sensitivity matrix computations are analysed.

When the measured state is used in eqn. **21** for the state p, the state equation will be of the form

$$\dot{p} = L_p p_m + L_\delta \delta \tag{34}$$

The measured p is appended to the measured control surface deflection δ, and hence in eqn. 34 the system matrix $A = 0$ and $B = [L_p, L_\delta]$. Hence $\phi = 1$ and $\psi = \Delta t$. Eqn. 34 in discrete form (using eqn. 24) is given by

$$p(k+1) = [1]p(k) + \Delta t[L_p \quad L_\delta]\begin{bmatrix} p_m(k) \\ \delta(k) \end{bmatrix} \tag{35}$$

As in the case of the derivations of eqns. 32 and 33, the cost function (eqn. 23) is to be mimmised with respect to the parameter vector $\Theta = [L_p, L_\delta]$. It is to be emphasised here that any change in the parameter L_p or L_δ causes the state p to change and hence in the subsequent expressions, the partial differentiation of the measured state with respect to the parameters is retained. The control surface deflection δ is treated independent of the parameters.

Applying eqn. 19 to eqn. 35 yields the following sensitivity equations:

$$\frac{\partial p(k+1)}{\partial L_p} = \phi\frac{\partial p(k)}{\partial L_p} + \Delta t\frac{\partial}{\partial L_p}[L_p \quad L_\delta]\begin{bmatrix} p_m(k) \\ \delta(k) \end{bmatrix} + \Delta t[L_p \quad L_\delta]\frac{\partial}{\partial L_p}\begin{bmatrix} p_m(k) \\ \delta(k) \end{bmatrix} \tag{36}$$

$$= \frac{\partial p(k)}{\partial L_p} + L_p\Delta t\frac{\partial p_m(k)}{\partial L_p} + \Delta t p_m(k) \tag{37}$$

$$\frac{\partial p(k+1)}{\partial L_\delta} = \frac{\partial p(k)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_m(k)}{\partial L_\delta} + \Delta t\delta(k) \tag{38}$$

The measured state p_m can be written as

$$p_m = p_t + p_n \tag{39}$$

where p_t is the true value of the roll rate p and p_n is the measurement noise in p. Substituting the above expression in eqns. 37 and 38, one obtains:

$$\frac{\partial p(k+1)}{\partial L_p} = \frac{\partial p(k)}{\partial L_p} + L_p\Delta t\frac{\partial p_t(k)}{\partial L_p} + L_p\Delta t\frac{\partial p_n(k)}{\partial L_p} + \Delta t p_t(k) + \Delta t p_n(k) \tag{40}$$

$$\frac{\partial p(k+1)}{\partial L_\delta} = \frac{\partial p(k)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_t(k)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_n(k)}{\partial L_\delta} + \Delta t\delta(k) \tag{41}$$

The first gradient is given by

$$\frac{\nabla E_S(\Theta)}{N-1} = \frac{1}{N-1}\begin{bmatrix} \sum_{k=1}^{N}(p_m(k)-p(k))\left[\frac{\partial p(k-1)}{\partial L_p} + L_p\Delta t\frac{\partial p_t(k-1)}{\partial L_p} + L_p\Delta t\frac{\partial p_n(k-1)}{\partial L_p} + \Delta t p_t(k-1) + \Delta t p_n(k-1)\right] \\ \sum_{k=1}^{N}(p_m(k)-p(k))\left[\frac{\partial p(k-1)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_t(k-1)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_n(k-1)}{\partial L_\delta} + \Delta t\delta(k-1)\right] \end{bmatrix} \tag{42}$$

In eqn. 42, since the measurement noise is independent of the parameters to be estimated, and it is assumed to be a white random process, the terms involving p_n drop out on average, in the statistical sense. Thus finally we have

$$\frac{\nabla E_S(\Theta)}{N-1} = \frac{1}{N-1}\begin{bmatrix} \sum_{k=1}^{N}(p_m(k)-p(k))\left[\frac{\partial p(k-1)}{\partial L_p} + L_p\Delta t\frac{\partial p_t(k-1)}{\partial L_p} + \Delta t p_t(k-1)\right] \\ \sum_{k=1}^{N}(p_m(k)-p(k))\left[\frac{\partial p(k-1)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_t(k-1)}{\partial L_\delta} + \Delta t\delta(k-1)\right] \end{bmatrix} \tag{43}$$

Next, in eqn. *32* the integrated state p is replaced by $p_t + p_i$ (only where appropriate in the bracketed terms), where p_i, denotes the error in integration due to incorrect initial conditions or any other related errors, to obtain (for the **OEM)**

$$\frac{\nabla E_O(\Theta)}{N-1} = \frac{1}{N-1}\begin{bmatrix} \sum_{k=1}^{N}(p_m(k)-p(k))\left[\frac{\partial p(k-1)}{\partial L_p} + L_p\Delta t\frac{\partial p_t(k-1)}{\partial L_p} + L_p\Delta t\frac{\partial p_i(k-1)}{\partial L_p} + \Delta t p_t(k-1) + \Delta t p_i(k-1)\right] \\ \sum_{k=1}^{N}(p_m(k)-p(k))\left[\frac{\partial p(k-1)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_t(k-1)}{\partial L_\delta} + L_p\Delta t\frac{\partial p_i(k-1)}{\partial L_\delta} + \Delta t\delta(k-1)\right] \end{bmatrix} \tag{44}$$

As the parameter estimation is done iteratively, the use of improved estimates in the integration makes the integration error p_i tend to zero as the iteration progresses (here the convergence of the algorithm is assumed). and hrncr the expression for the first gradient. eqn. 44, for tlie OEM becomes

$$\frac{\nabla E_O(\Theta)}{N-1} \tag{45}$$

$$= \frac{1}{N-1} \left| \begin{array}{l} \sum_{k=1}^{N}(p_m(k) - p(k)) \left[\frac{\partial p(k-1)}{\partial L_p} + L_p \Delta t \frac{\partial p_t(k-1)}{\partial L_p} + \Delta t p_t(k-1)\right] \\ \sum_{k=1}^{N}(p_m(k) - p(k)) \left[\frac{\partial p(k-1)}{\partial L_\delta} + L_p \Delta t \frac{\partial p_t(k-1)}{\partial L_\delta} + \Delta t \delta(k-1)\right] \end{array} \right|$$

Comparison of sensitivity eqn. 43 for the SOEM ith eqn. 45 for the OEM. reveals that the asymptotic behaviour of the SOEM is similar to that of the OEM. However, the OEM does not work directly for unstable systems, because the numerical integration diverges owing to the unstable system. In case of SOEMs, since the measured states (obtained from the unstable plant operating in a closed loop) are stable, their use in the estimation process tries to prevent this divergence, and at the same time enables parameter estimation of the basic unstable plant directly, in a manner similar to that of the **OEM** for the stable system. Similar analysis (not included in the paper) carried out for second-order longitudinal short-period dynamics has also established the validity of the procedure. Thus, in essence, the asymptotic analysis has shown that the **SOEMs,** when applied to unstable systems: would behave in an almost similar manner as the **OEM** would behave when applied to a stable system.

Intuitively, to substantiate the explanation of the working of SOEMs. we can consider a second-order unstable system as follows:

$$\dot{x}_{1_i} = a_{11}x_{1_i} + a_{12}x_{2_i} + b_1 u_1 \tag{46}$$

$$\dot{x}_{2_i} = a_{21}x_{1_i} + a_{12}x_{2_i} + b_2 u_1 \tag{47}$$

The subscript *i* stands for integrated state. When these equations are integrated, the unstable system parameters cause numerical divergence of the states. Assuming that the parameter that causes divergence is a_{21}, if the state x_{1i} is replaced by measured x_{1m}, we have the following state equations:

$$\dot{x}_{1_i} = a_{11}x_{1_i} + a_{12}x_{2_i} + b_1 u_1 \tag{48}$$

$$\dot{x}_{2_i} = a_{11}x_{1_m} + a_{12}x_{2_i} + b_2 u_1 \tag{49}$$

When eqns. 48 and 49 are integrated. by the use of x_{1_m}, the divergence of x_{2i} in eqn. 49 is arrested, and hence that in eqn. 4X is also arrested. Thus, the use of a measured state in the integration procedure effectively stabilises the output-error cost function (eqn. 4). In general, the parameters which cause numerical instability are related to the so-called offending states (q, α), which, in practice. are measurable. In particular. these and related states are used as flight-control system variables.

6 Numerical validation

Example 1: Short period data of BEAVER aircraft are simulated. The static stability parameter M_w is adjusted to give a system with a time to double of Is. The data is generated with a doublet input to the pilot stick with a sampling time of 0.05 s [10–15].

State equations:

$$\dot{w} = Z_w w + (u_0 + Z_q)q + Z_{\delta_e}\delta_e \tag{50}$$

$$\dot{q} = M_w w + M_q q + M_{\delta_e}\delta_e \tag{51}$$

Observation **equations:**

$$A_z = Z_w w + Z_q q + Z_{\delta_e}\delta_e$$

$$w = w$$

$$q = q \tag{52}$$

where w is the vertical velocity, u_0 the stationary forward speed, q the pitch rate, A_z the vertical acceleration and δ_e is the elevator deflection. Since for M_w with a positive numerical value the system is unstable, vertical velocity is fed back with a gain k to stabilise the system as follows:

$$\delta_e = \delta_p + kw \tag{53}$$

where δ_p denotes the pilot input. Various sets of data are generated by varying the gain k to study the effect of gain k on the estimated parameters. The direct identification between δ_e and output measurements is attempted. Figs. 1 and 2 show the time-history match when the OEM is used to analyse the data. The numerical divergence caused duc to integration of the inherently unstable plant is clear from the Figure.

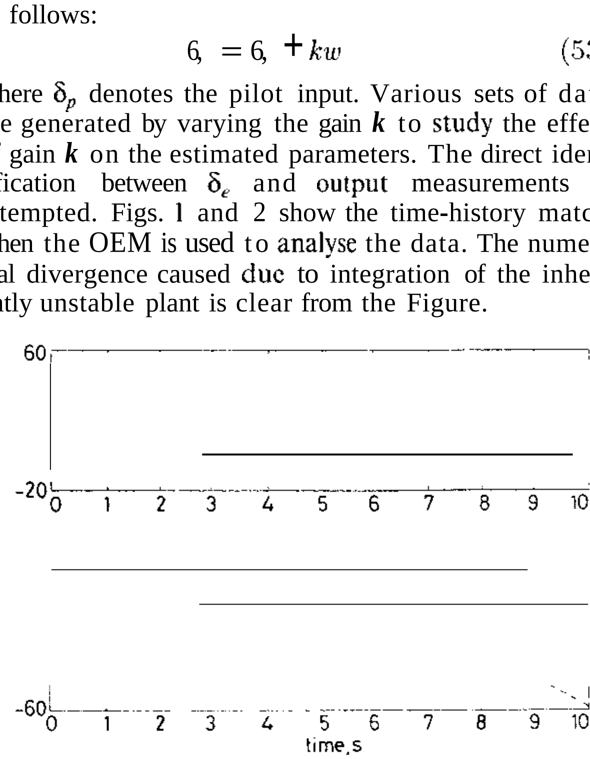

Fig. 1 *Time-history match using the OEM (example 1)*
—— measured -·-·- estimated

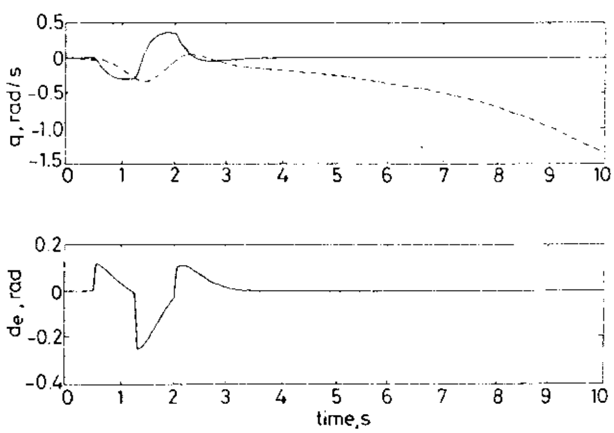

Fig. 2 *Time-history match using the OEM (example 1)*
—— measured -·-·- estimated

The EDOEM and the RAOEM are next applied to the same data. Figs. 3 and 4 show the time-history match when the RAOEM is applied. The time-history match is very good, indicating the benefit of the use of measured states in the numerical integration procedure.

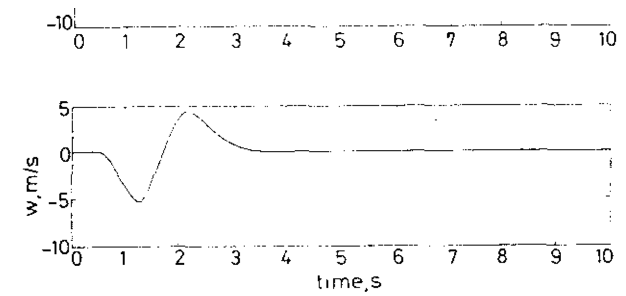

Fig.3 *Time-history match using the RAOEM (example 1)*
—— measured –·–·– estimated

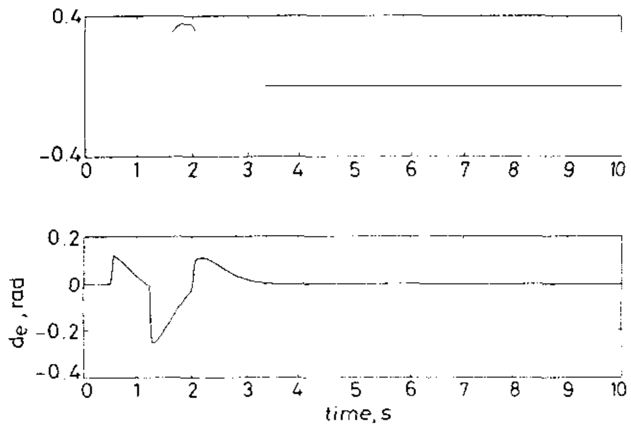

Fig.4 *Time-history match using the RAOEM (example 1)*
—— measured –·–·– estimated

In the RAOEM for this data. measured states w and q are used in the equation for q, eqn. 51, since it is known that. in this case. the derivatives M_w and M_q contribute to the instability. The estimated derivatives for increasing feedback gains from k = *0.025* to k = *0.25* are given in Table 1. **All** the parameter estimates are very close to the true values when the gains are small.

Table 1: Parameter estimates using the RAOEM (example 1)

	Gain $k \rightarrow$	0.025	0.05	0.25
Parameter	true value↓			
Z_w	−1.4249	−1.4166	−1.4172	−1.3947
Z_q	−1.4768	−1.4958	−1.4702	−1.2347
$Z_{\delta e}$	−6.2632	−6.5408	−6.5666	−6.8637
M_w	0.2163	0.2173	0.2183	0.2267
M_q	−3.7067	−3.7266	−3.7469	−3.9077
$M_{\delta e}$	−12.784	−12.8568	−12.8568	−13.1809
L1%	—	1.0397	1.6647	8.1598
L2%	—	0.7447	1.2164	4.7577

The EDOEM is also used to analyse the data by decoupling the state equations using measured states for the off-diagonal elements. These results are given in Table 2, from which it is clear that, as in the case of the RAOEM, when the gains are small (k = *0.025* and k= 0.05), the estimates are close to the true values. The results indicate that. in the absence of measurement noise and with small feedback gains. the EDOEM and the RAOEM can he used successfully for estimation of aerodynamic derivatives of unstable/augmented systems.

Table 2 Parameter estimates using the EDOEM (example 1)

	Gain $k \rightarrow$	0.025	0.05	0.25
Parameter	true value↓			
Z_w	−1.4249	−1.4286	−1.4299	−1.4375
Z_q	−1.4768	−1.3708	−1.3363	−1.1165
$Z_{\delta e}$	−6.2632	−6.2596	−6.2647	−6.3884
M_w	0.2163	0.2178	0.2188	0.2275
M_q	−3.7067	−3.7380	−3.7580	−3.9464
$M_{\delta e}$	−12.784	−12.8490	−12.8884	−13.2163
L1%	—	0.8159	1.1797	4.5660
L2%	—	0.4960	0.7054	2.4142

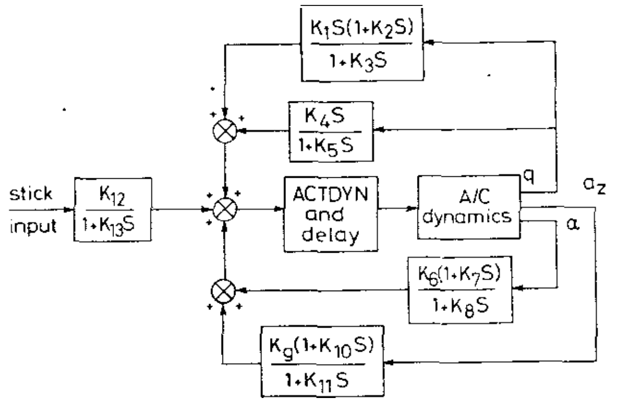

Fig.5 *Block diagram of simulated closed-loop system*

Example 2: **A** typical fourth-order longitudinal **FBWCS** system, shown in Fig. 5, is simulated at a nominal flight condition. The dynamics of the basic aircraft are [13] given by

State equations:

$$\begin{bmatrix} \dot\alpha \\ \dot q \\ \dot\theta \\ \dot\nu/\nu_0 \end{bmatrix} = \begin{bmatrix} Z_\alpha & 1 & 0 & Z_\nu \\ M_\alpha & M_q & 0 & M_\nu \\ 0 & 1 & 0 & 0 \\ X_\alpha & 0 & X_\theta & X_\nu \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \\ \nu/\nu_0 \end{bmatrix} + \begin{bmatrix} Z_{\delta_e} \\ M_{\delta_e} \\ 0 \\ X_{\delta_e} \end{bmatrix} \delta_e \tag{54}$$

Measurement equations:

$$\begin{bmatrix} \alpha \\ q \\ a_x \\ a_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ C_{31} & 0 & 0 & C_{34} \\ C_{41} & 0 & 0 & C_{44} \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \\ \nu/\nu_0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ D_{31} \\ D_{41} \end{bmatrix} \delta_e \tag{55}$$

where the $Z_{()}$, $X_{()}$, $M_{()}$, $C_{()}$ and $D_{()}$ are the aerodynamic derivatives to **be** estimated. The OEM, the EDOEM and the **RAOEM** are applied to this data set. When the OEM is used, the algorithm tends to converge (in the sense of the determinant of the covariance matrix R). but the time-history match is not satisfactory, as seen in Fig. 6. Table **3** lists estimates of all the parameters.

For the RAOEM. measured states (α_m, q_m) are used in the q equation, since the derivatives cpntributing to the divergence in the longitudinal dynamics are a part of this equation. leading to the following formulation of the state equations:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{\nu}/\nu_0 \end{bmatrix} = \begin{bmatrix} Z_\alpha & 1 & 0 & 0 \\ 0 & 0 & 0 & M_\nu \\ 0 & 0 & 1 & 0 \\ X_\alpha & 0 & X_\theta & X_\nu \end{bmatrix} \begin{bmatrix} \alpha \\ q \\ \theta \\ \nu/\nu_0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & Z_{\delta_e} \\ M_\alpha & M_q & M_{\delta_e} \\ 0 & 0 & 0 \\ 0 & 0 & X_{\delta_e} \end{bmatrix} \begin{bmatrix} \alpha_m \\ q_m \\ \delta_e \end{bmatrix} \quad (56)$$

For applying the EDOEM, the A matrix is decoupled into two parts containing the diagonal elements and off-diagonal elements as follows:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{q} \\ \dot{\theta} \\ \dot{\nu}/\nu_0 \end{bmatrix} = \begin{bmatrix} Z_\alpha & 0 & 0 & 0 \\ 0 & M_q & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & X_\nu \end{bmatrix} \begin{bmatrix} a \\ q \\ \theta \\ \nu/\nu_0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & Z_{\delta_e} \\ M_\alpha & 0 & M_{\delta_e} \\ 0 & 0 & 0 \\ X_\alpha & 0 & X_{\delta_e} \end{bmatrix} \begin{bmatrix} \alpha_m \\ q_m \\ \delta_e \end{bmatrix} \quad (57)$$

The estimates by these methods are shown in Table 3, and the time-history match is shown in Fig. 7. The match is very good, indicating the advantage of using measured states for avoiding numerical divergence problems.

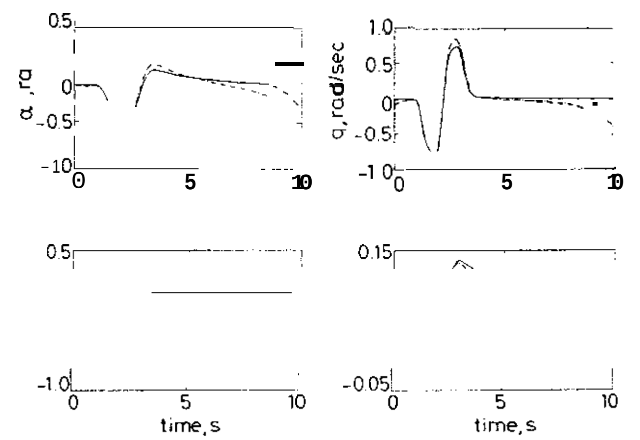

Fig.6 *Time-history match using the OEM*
——— measured - - - - estimated

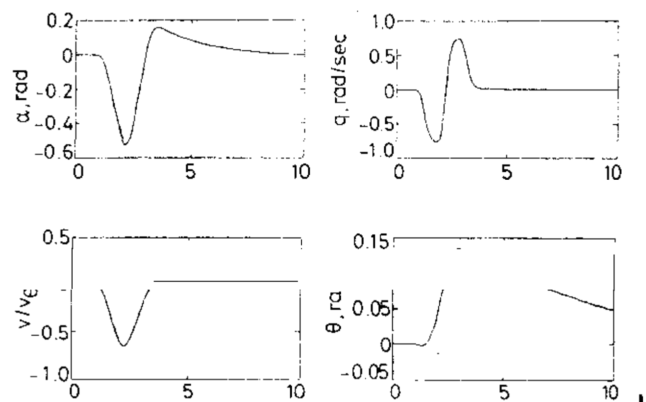

Fig.7 *Time-history match using the EDOEM*
——— measured - - - - estimated

Thus it is clear that both the KAOEM and the EDOEM perform well. and enable parameter estimation of unstable/augmented systems in the presence of feedback for unstable basic plant, despite the large number of parameters that are estimated. The estimates of the significant flight mechanical parameters. M_α, M_q, $M_{\delta e}$, are close to their true values. **Also,** the numerical performance of the SOEMs is much better than that of the OEM (when both the techniques are applied to unstable/augmented systems) in terms of $L1$ and $L2$ norms (where norm (Θ, P) = sum [absolute $(\Theta)^{P}$ $^{1/P}$; P = 1, 2). Thus, the parameter estimation using SOEMs for a system with a large number of control feedback loops and parameters has been successfully achieved in this paper.

Table 3: Parameter estimates using the RAOEM and the EDOEM (example 2)

Parameter	True values	OEM	RAOEM	EDOEM
Z_α/v_0	-0.4432	-0.4432	-0.4123	-0.4123
$Z_{\delta e}$	-0.1499	-0.5324	-0.1929	-0.1883
Z_v/v_0	-0.1955	-0.1302	-0.1599	-0.1601
M_α	1.3316	1.4009	1.3459	1.3488
M_q	-0.4717	4.4611	-0.4748	-0.4714
$M_{\delta e}$	-4.8267	-5.1973	-4.8614	-4.8656
X_α	-0.0965	-0.0853	-0.0943	-0.0959
X_v/v_0	-0.0443	-0.1081	-0.0477	-0.1019
$X_{\delta e}$	-0.0429	4.0527	-0.0449	-0.0459,
X_θ	-0.1018	-0.0204	-0.1023	-0.0443
M_v/v_0	0.0226	-1.7899	0.0161	0.0032
C31	0.9179	1.1512	0.9094	0.9088
C34	0.4879	-0.0204	0.0011	0.0070
C41	-41.387	-47.0086	-41.003	-40.9754
C44	-19.271	-42.5008	-19.8553	-19.8759
L1%	—	46.52	2.34	2.58
L2%	—	52.19	1.85	1.92

7 Conclusions

In this paper, asymptotic analysis of stabilised output-error methods for parameter estimation of unstable/augmented systems has been presented. The methods use measured states, to avoid numerical divergence caused by integration of the state equations in the parameter-estimation procedure. The methods are validated using simulated data of second-order short period and fourth order longitudinal dynamics of an aircraft.

8 Acknowledgment

The authors are grateful to Dr. S. Srinathkumar (Head. FMCD, and Deputy Director, NAL) for moral support and useful technical discussions during the course of this work.

9 References

1 SOKENSON. H.W.: 'Parameter estimation – principles and problem' (Marcel Dekker. New Yark. 1980)
2 SINHA, N.K., and KUTSZA. B.: 'Modeling and identification of multivariable dynamic systems' (Van Nostrand, New York, 1983)
3 EYKHOFF, P.: 'System identification – parameter and slate estimation' (John Wiley and Sons. 1977)
4 GOPALRATNAM G., and JATEGAONKAK. K.V.: 'Some results of ATTAS flight data analysis using maximum likelihood parameter estimation method. DLR FB 91–04, 1991

5 GOPALRATNAM, G., PARAMESWARAN, V., RAOL, J.R., and BALAKRISHNA, S.: 'Estimation of aircraft performance and stability characteristics by dynamic manoeuvres using maximum likelihood estimation techniques'. Pre-prints of 9th IFAC/IFORS Symposium on *Identification and system parameter estimation*, Budapest, Hungary, 8-12 July 1991

6 PARAMESWARAN, V., and PLAETSCHKE, E.: 'Flight path reconstruction using extended Kalman filtering techniques', DLR-FB 90-41; also as NAL PD FC 9103

7 MAINE, R.E., and ILIFF, K.W.: 'Application of parameter estimation to aircraft stability and control – the output error approach', NASA RP 1168, 1986

8 MAINE, R.E., and MURRAY, J.E.: 'Application of parameter estimation to highly unstable aircraft', *J. Guid. Control Dyn.*, 1988, **11**, (3), pp. 213–219

9 PREISSLER, H., and SCHAUFELE, H.: 'Equation decoupling - a new approach to the aerodynamic identification of unstable aircraft', *J. Aircr.*, 1991, **28**, (2), pp. 146–150

10 JATEGAONKAR, R.V.: 'Parameter estimation methods for unstable aircraft', DLR Mitt-93-14, Dec 1993, (also as NAL SP 9324)

11 MENDEL J.M.: 'Discrete techniques for parameter estimation - the equation error formulation' (Marcel Dekker Inc., New York, 1973)

12 RAOL, J.R., and JATEGAONKAR, R.V.: 'Artificial neural networks for aerodynamic modelling', DLR IB 111 94/41, October 1994

13 GOPALRATNAM, G., and RAOL, J.R.: 'Estimation of parameters of unstable, and augmented aircraft system using recursive mixed estimation technique', *J. Inst. Eng. Aerosp. Div.*, 1995, **76**, pp. 15–22

14 KLEIN V.: 'Estimation of aircraft aerodynamic parameters from flight data', *Prog. Aerosp. Sci.*, 1989, **26**, pp. 1–77

15 PLAETSCHKE, E., MULDER, J.A., and BREEMAN, J.H.: 'Results of **Braver** aircraft parameter identification', DFVLR-FB 83-10, March 1983