A HIGH PRECISION TRIANGULAR LAMINATED ANISOTROPIC CYLINDRICAL SHELL FINITE ELEMENT

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Abstract—The stiffness matrix for a high precision triangular laminated anisotropic cylindrical shell finite element has been formulated and coded into a composite structural analysis program. The versatility of the element's formulation enables its use in the analysis of multilayered composite plate and cylindrical shell type structures taking into account actual lamination parameters. The example applications presented demonstrated that accurate predictions of stresses as well as displacements are obtained with modest number of elements.

INTRODUCTION

Multilayered composite materials are increasingly used in the construction of plate and shell type structures for various industrial and aero-space applications, creating a parallel need for structural analysis procedures which reflect the laminated anisotropic characteristics of such materials. Although the basic equations which govern the elastic behaviour of laminated anisotropic plate and shell structures can be derived[1], analytical solutions of these complex equations, as they pertain to design and analysis, are not available. The analysis of composite structures is an area which calls for an extensive use of digital computers and numerical methods like the finite element method.

The finite element method has proved to be an extremely powerful tool for the solution of problems involving complex geometries, arbitrary loadings and rather general material properties. Considerable advances have been made in the development and application of the finite element method to plate and shell type structures. Much of the work is reviewed by Gallagher[2,3]. According to these reviews, the analysis of thin shells is one of the more difficult problems that has been attempted with the finite element method. The requirements upon valid minimum potential energy solutions in thin shell finite element analysis are extremely difficult to satisfy, and the satisfactory formulations are therefore relatively complicated. Further, for tackling problems involving description of arbitrary boundaries triangular and quadrilateral elements are obviously best suited. The most reliable and sophisticated triangular shell element formulations are those due to Dupuis[4], Cowper et al.[5], Argyris and Scharff[6] and Dawe[7]. The formulation due to Cowper et al.[5] has also been converted into a triangular cylindrical shell finite element by Lindberg and Olson[8]. The salient features of this element are: (i) The element is fully conforming, of the displacement type and of arbitrary triangular shape, (ii) Uses higher order interpolation polynomials and has 36 degrees of freedom, (iii) Transverse displacement functions are complete cubic polynomials to yield a cubic variation of normal stresses within the element, (iv) Tangential displacement functions are complete cubic polynomials to yield a quadratic variation of stresses within the element, (v) Gives accurate predictions of displacements as well as stresses, (vi) Satisfies sufficient conditions to guarantee rapid convergence, (vii) Permits reliable and direct evaluation of stresses at nodes, (viii) The element stiffness matrix is formulated in exact closed form, (ix) Results in a smaller overall problem size and (x) The versatility of the formulation enables its use in the analysis of multilayered composite plate and cylindrical shell type structures subjected to in-place and bending loads. The applications presented encompassing plate elements in plane stress and in flexure and circular cylindrical shells clearly indicate the usefulness of the element in composite structural analysis.

FORMULATION

The geometry of an arbitrary triangular cylindrical shell element is shown in Fig. 1. The global cylindrical coordinate system is \( X, Y \) and \( \xi, \eta \) are taken as local coordinates for the element. \( R \) is the radius of curvature of the reference surface of the shell and \( t \) is its wall thickness. The dimensions \( a, b, c \) of the triangle \( 1, 2, 3 \) and the rotation angle \( \phi \) are easily derived in terms of the global coordinates of the vertices [9].

The strain energy of a thin cylindrical shell in global coordinates is given by

\[
U = \frac{1}{2} \int \left( [N]^T [e]^p + [M]^T [x] \right) ds dy
\]

where

\[
[N]^T = \begin{bmatrix} N_x N_y N_{xy} X_{xy} \end{bmatrix}
\]

are the membrane forces

\[
[M]^T = \begin{bmatrix} M_x M_y M_{xy} X_{xy} \end{bmatrix}
\]

are the bending stress resultants

\[
[e]^p = \begin{bmatrix} e_x e_y e_{xy} \end{bmatrix}
\]

are the reference surface strains and

\[
[x] = \begin{bmatrix} X_x X_y X_{xy} \end{bmatrix}
\]

are the bending curvatures.

For an arbitrarily laminated anisotropic shell, the con-
when radially outward. Note that subscript and (5) denote differentiation.

Substituting for \( \varepsilon^P \) and \( \chi \) from eqns (4) possible to express the strain energy given matrix form as

\[
U = \frac{1}{2} \int \int [H_1^t \cdot [L] \cdot [H_1]] \, dx \, dy
\]

where

\[
[H_1] = \begin{bmatrix} [R_1] \end{bmatrix}
\]

and the strain energy matrix \([L]\) is given in Table I. If \( U, V \) denote reference surface displacement local coordinate system \((\xi, \eta)\), the transformation to local to global coordinates is

\[
[d] = \begin{bmatrix} R_1^t \end{bmatrix} \cdot \begin{bmatrix} [d] \end{bmatrix}
\]

Substituting eqn (7) in eqn (6), the strain energy of the element is given by

\[
U = \frac{1}{2} \int \int [H_1^t \cdot [L] \cdot [H_1]] \, d\xi \, d\eta
\]

with the area integration to be carried out over the triangle

The assumed displacement functions for the element are

\[
\bar{U} = \sum_{i=1}^{m} a_i \varepsilon^m \eta^n
\]

\[
\bar{V} = \sum_{i=1}^{n} b_i \varepsilon^n \eta^m
\]

\[
\bar{W} = \sum_{i=1}^{s} c_i \varepsilon^s \eta^n
\]

where \( m, n, p, r, s \) are just integers.

The element stiffness matrix is obtained from calculation of its strain energy. The displacement functions of eqns (9) are substituted into eqn (8) and carried out to yield

\[
U = \frac{1}{2} [a] \cdot [k] \cdot [a]
\]

where

\[
[a] = \begin{bmatrix} A_{11} & A_{12} & A_{13} + \frac{2B_1}{R_c} \\ A_{21} & A_{22} & A_{23} + \frac{2B_2}{R_c} \\ A_{31} & A_{32} & A_{33} + \frac{4B_1 + 2D_1}{R_c} \end{bmatrix}
\]

\[
[k] = \begin{bmatrix} 2H_1 & R_c \\ 2H_2 & R_c \\ 2H_3 & R_c \end{bmatrix}
\]

<table>
<thead>
<tr>
<th>A_{11}</th>
<th>A_{12}</th>
<th>A_{13} + \frac{2B_1}{R_c}</th>
</tr>
</thead>
<tbody>
<tr>
<td>A_{21}</td>
<td>A_{22}</td>
<td>A_{23} + \frac{2B_2}{R_c}</td>
</tr>
<tr>
<td>A_{31}</td>
<td>A_{32}</td>
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</tr>
</tbody>
</table>

Table I. Strain energy matrix \([L]\)
A high precision triangular laminated anisotropic cylindrical shell finite element

Table 2. Rotation matrix \([\mathbf{R}]\)

\[
\begin{bmatrix}
CC & -SC & -SS & 0 & 0 & 0 \\
SC & CC & -SS & -SC & 0 & 0 \\
-SS & SC & CC & -SC & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

where \(CC = \cos^2 \theta\), \(SS = \sin^2 \theta\), \(SC = \sin \theta \cos \theta\)

where \([a]\) is a 40-column vector of the polynomial coefficients \(a_i\). The entries of the stiffness matrix \([k]\) are determined in closed form and are given in the Appendix.

The generalized displacements in global coordinates chosen for the element are

\[
\begin{align*}
[W]^T &= (U_1, U_2, U_3, V_1, V_2, V_3, W_1, W_2, W_3) \\
W_{XX}, W_{XY}, W_{YY}, U_2, \ldots, U_n, \ldots, W_{YY}.
\end{align*}
\]

Further steps to be followed in the deviation of the 36 x 36 stiffness matrix relating the generalized displacements \([W]\) and the corresponding generalized forces, and consistent load vector corresponding to uniform pressure load are very similar to those for the shallow shell element [5].

COMPUTER PROGRAM

The stiffness matrix and consistent load vector for the high precision triangular laminated anisotropic cylindrical shell finite element has been coded into a composite structural analysis program. Distinct membrane and flexural stiffnesses are retained permitting analysis of composite structures subjected to inplane and bending loads. The resulting force-displacement equations are solved using Choleski square root decomposition method for fixed bandwidth problems. Stress resultants and individual layer stresses are directly calculated at the nodes using the generalized displacements, constitutive relations and appropriate transformation equations. With basic lamina elastic constants and laminate details as input, the program evaluates the laminate stiffnesses for each element. It is therefore possible to analyse laminates of varying thickness taking into account the actual lamination parameters. The computer output consists of generalized displacements, stress resultants, membrane and bending strains, individual layer stresses with reference to both laminate and lamina reference axes at each node. With this information it is possible to make strength prediction of practical composite structures using appropriate failure criteria [1].

APPLICATIONS

The versatility of the element formulation enables its use in the analysis of laminated anisotropic flat plates in either membrane or flexural behaviour and for coupled membrane-bending behaviour of circular cylindrical shells. The following example applications illustrate the usefulness of the element in composite structural analysis. All calculations are carried out on an IBM 370/155 machine using double precision arithmetic.

Boron-epoxy composite laminate with a circular hole under axial tension

In Fig. 2 is shown a boron-epoxy composite panel consisting of nine plies of [0/±45/0/90], laid up with a central circular hole under uniaxial tension. Symmetry of the loading, geometry and material properties make the analysis of only one quarter of the panel sufficient. The finite element idealization of the quarter panel illustrated in Fig. 3 uses 51 elements with 37 nodes resulting in 186 degrees of freedom. Typical stress distributions calculated at the nodes are shown in Figs. 4 and 5 along with the results of anisotropic elasticity solution to the problem of an infinite plate with a circular opening[11]. The present results appear to have converged quite rapidly. The predicted tensile and compressive stress and strain concentration factors given in Table 3 together with results of anisotropic elasticity solution for the plate clearly demonstrate that the element gives excellent accuracy.

An identical problem has been analysed in Ref. [12] using linear-strain triangular elements (twelve degrees of freedom). One hundred and forty one elements with 320 nodes (598 degrees of freedom) were used to describe

Fig. 2. Boron-epoxy laminate with a circular hole under axial tension.

Fig. 3. 5 x 5 Finite element grid.
the quarter panel. The predictions are also given in Table 3 for comparison. The fact that the present formulation yields better accuracy with about one-third the problem size demonstrate the superiority of the element.

The problem has also been analysed experimentally using straingages, photoelastic coatings and moire techniques in Ref. (12). Average predictions are given in Table 3. The close agreement between the present results and experimental data readily verifies the adequacy of the assumed laminated anisotropic model in the analysis as a first approximation to the actual multilayered composite material behaviour.

**Clamped glass-epoxy composite square plate subjected to uniform pressure load.** Figure 6 shows a glass-epoxy composite square plate with clamped edges subjected to uniform pressure load. Symmetry allows the analysis to be limited to one quarter of the plate. Arrangement of the elements in a quadrant of the plate are shown in the figure. The predictions are also given in Table 3 for comparison. The fact that the present formulation yields better accuracy with about one-third the problem size demonstrates the superiority of the element.

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**Clamped glass-epoxy composite square plate subjected to uniform pressure load.** Figure 6 shows a glass-epoxy composite square plate with clamped edges subjected to uniform pressure load. Symmetry allows the analysis to be limited to one quarter of the plate. Arrangement of the elements in a quadrant of the plate are shown in the figure. The predictions are also given in Table 3 for comparison. The fact that the present formulation yields better accuracy with about one-third the problem size demonstrates the superiority of the element.

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**Table 3. Boron-epoxy laminate with a circular hole under axial tension**

<table>
<thead>
<tr>
<th>No. of unknowns</th>
<th>Stress concentrat</th>
<th>Steam concentrat</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>factors at 0°, 90°</td>
<td>factors at 0°, 90°</td>
</tr>
<tr>
<td>186</td>
<td>0.6957</td>
<td>1.4982</td>
</tr>
<tr>
<td>598</td>
<td>0.6829</td>
<td>1.46</td>
</tr>
<tr>
<td>1.45</td>
<td></td>
<td>3.45</td>
</tr>
<tr>
<td>1.60</td>
<td></td>
<td>3.34</td>
</tr>
<tr>
<td>1.46</td>
<td></td>
<td>3.34</td>
</tr>
<tr>
<td>1.5</td>
<td></td>
<td>3.34</td>
</tr>
<tr>
<td>3.80</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Comparison of results
A high precision triangular laminated anisotropic cylindrical shell finite element

MATERIAL PROPERTIES

\[
\begin{align*}
E_{11} &= 7.5 \times 10^6 \text{ psi} \\
E_{22} &= 2.0 \times 10^6 \text{ psi} \\
G_{12} &= 1.25 \times 10^6 \text{ psi} \\
v_{12} &= 0.25 \\
\frac{h}{l} &= 0.1 \text{ in}
\end{align*}
\]

Fig. 6. Finite element grid for clamped glass-epoxy composite square plate.

and deflections is not possible due to inherent approximations in the available solution in Ref. [13]. (The solution in [13] when specialized to isotropic case shows an error of about 6.3% for maximum deflection and unacceptable accuracy for maximum moments.)

An orthotropic cylindrical shell with a circular hole subjected to axial tension. The problem illustrated in Fig. 9 is analysed to evaluate the coupled membrane-bending behaviour of the cylindrical shell form of the element. Taking advantage of symmetry, only one quarter of the shell needs to be analysed. Choosing the 5 x 5 finite element grid illustrated in Fig. 3, 51 elements with 37 nodes (367 degrees of freedom) are used to model the quarter of the shell. The adequacy of the chosen grid to solve the present problem was checked by solving an otherwise identical isotropic shell problem. Typical results are given in Table 5 along with continuum solutions [15] for comparison to demonstrate the accuracy achieved with the rather coarse mesh used.

Predicted stress distributions for both isotropic and orthotropic shells are shown in Figs. 10 and 11 along with the results of anisotropic elasticity solution to the problem of an infinite plate with a circular hole [11]. The predicted stress distributions are fairly smooth and appear to have converged as well. Near the hole the coupled membrane-bending behaviour appears to be
Table 4. Clamped glass-epoxy composite square plate subjected to uniform pressure load—comparison of results

<table>
<thead>
<tr>
<th>Maximum deflection</th>
<th>Bending moments at X = d/2, Y = 0</th>
<th>Bending moments at X = 0, Y = d/2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Present results</td>
<td>0.00246</td>
<td>0.00424</td>
</tr>
<tr>
<td>Approximate</td>
<td>0.00243</td>
<td>0.00413</td>
</tr>
<tr>
<td>analytical solution Ref. [10]</td>
<td>0.00243</td>
<td>0.00414</td>
</tr>
<tr>
<td></td>
<td>0.00243</td>
<td>0.00414</td>
</tr>
</tbody>
</table>

Table 5. Cylindrical shell with a circular hole under axial tension—comparison of results

| Membrane stress ratio (σ_m/|σ_t|) | Bending stress ratio (σ_b/|σ_t|) |
|----------------------------|---------------------------------|---------------------------------|
| Orthotropic, shell         |                                 |                                 |
| Present results            | 0.90164                         | 4.9920                          |
| Orthotropic plate          |                                 |                                 |
| Solution Ref. [12]         | 0.4813                          | 4.8594                          |
| Isotropic shell             |                                 |                                 |
| Present results            | 1.26018                         | 3.70422                         |
| Isotropic shell results from Ref. [11] | 1.25   | 1.6  |

† At inner surface of the shell

CONCLUSIONS
The stiffness matrix for a high precision triangular finite element has been formulated, and coded into a finite element analysis program. The versatility of the formulation enables its use in the analysis of composite plate and cylindrical shell structures taking into account actual lamination parameters obtained with the use of the sub element procedures in a design exercise. The ultimate objective of any analysis is to predict structural integrity of the multi-layered structures taking into account actual lamination parameters. The results of the present finite element analysis are compared with the experimental results. The finite element results are found to be in good agreement with the experimental results.

Fig. 9. Cylindrical shell with a circular hole under axial tension.
A high precision triangular laminated anisotropic cylindrical shell finite element

REFERENCES
15. P. Van Dyke, Stresses about a circular hole in a cylindrical shell. AIAA J., 3(9), 1733-1742 (1965).

APPENDIX
Stiffness Matrix \([k]\)

\[
k_{ij} = L_{ij}m_{ij}F(m_{ij}+m_{ij}-2, n_{ij}+n_{ij})
\]

\[
definition
\]

Fig. 10. Axial stress distribution along Y-axis.

Fig. 11. Tangential stress distribution along the hole boundary.
\[ + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{ij} b_{jk} c_{ik} \cdot F(i, j, k) \cdot (i + 2, j + 2, k + 2) \]

where

\[ F(i, j, k) = \frac{\eta_i \eta_j \eta_k}{(\eta_i + 1)(\eta_j + 1)(\eta_k + 1)} \cdot \frac{b_{ij} b_{jk} c_{ik}}{(i + 1)(j + 1)(k + 1)} \]

\[ \eta_i, \eta_j, \eta_k = 0 \text{ for } i > 10 \]

\[ b_{ij}, b_{jk}, c_{ik} = 0 \text{ for } i > 20 \]

\[ \eta_i, \eta_j, \eta_k = 0 \text{ for } i > 21 \]

\[ \eta_i, \eta_j, \eta_k = 1 \text{ for } i < 21 \]