Low-order parabolic theory for 2D boundary-layer stability

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We formulate here a lowest order parabolic (LOP) theory for investigating the stability of two-dimensional spatially developing boundary layer flows. Adapting a transformation earlier proposed by the authors, and including terms of order $R^{-1/2}$ where $R$ is the local boundary-layer thickness Reynolds number, we derive a minimal composite equation that contains only those terms necessary to describe the dynamics of the disturbance velocity field in the bulk of the flow as well as in the critical and wall layers. This equation completes a hierarchy of three equations, with an ordinary differential equation correct to $R^{-1/2}$ (similar to but different from the Orr-Sommerfeld) at one end, and a "full" nonparallel equation nominally correct to $R^{-1}$ at the other (although the latter can legitimately claim higher accuracy only when the mean flow in the boundary layer is computed using higher order theory). The LOP equation is shown to give results close to the full nonparallel theory, and is the highest-order stability theory that is justifiable with the lowest-order mean velocity profiles for the boundary layer. © 1999 American Institute of Physics.

I. INTRODUCTION

The downstream growth of a boundary layer of characteristic thickness $\theta(x)$ can be written as

$$\frac{d\theta}{dx} \sim O\left(\frac{1}{R}\right),$$

where $A$ is the streamwise coordinate and $R$ the Reynolds number based on boundary-layer thickness. At high Reynolds numbers, this variation with $x$ is small in comparison with variations of other physical quantities like the velocity with respect to the normal direction $y$. It has therefore been long considered a good first approximation to take the boundary layer as locally parallel in stability analyses, and apply the traditional Orr-Sommerfeld (OS) equation. Several penetrating studies (e.g., Gaster, Bertolotti et al.) sought to improve on the parallel-flow approximation by taking account of the spatial development. In most of these, the OS equation was accepted as being in some sense the lowest-order stability equation, with nonparallel effects adding higher-order $O\left(R^{-1}\right)$ corrections to it. To lend strength to this line of thought, several nonparallel formulations derived from first principles, such as Bertolotti, Herbert, and Späth (hereafter BHS) and Govindarajan and Narasimha (hereafter GN95), gave rise to more elaborate stability equations which contain the entire OS equation and several additional terms with $R^{-1}$ as factor. Some of these higher-order terms contain derivatives in the streamwise direction, which means that the stability equations are no longer ordinary differential equations like the Orr-Sommerfeld. Although the apparent difference between the parallel and nonparallel formulations is only in the higher-order terms, the implicit assumption that the stability of a boundary layer is not strongly affected by its growth was (for various reasons discussed below) in need of a critical re-examination, which has recently been undertaken by Govindarajan and Narasimha (hereafter GN97).

Considering similarity solutions of the boundary-layer equations, and adopting a coordinate transformation inspired by the similarity variables, they showed that the spatial development of the flow affects stability at an order considerably lower than $O\left(R^{-1}\right)$. This does not present any contradiction, since nonparallel effects are contained, not merely in the additional higher-order terms emerging from various formulations, but very substantially in the lower-order terms themselves. They showed that a "lowest-order" stability equation, which contained all effects of $O\left(R^{-1}\right)$ (and up to but not including $O\left(R^{-2}\right)$), produces stability results in good qualitative agreement with the results of the full nonparallel analysis. This equation is nonetheless an ordinary differential equation; the streamwise dependence is partly explicit, since $R \equiv R(x)$, but is also imbedded into it by the particular coordinate transformation used.

In nonparallel stability analyses which include terms of $O\left(R^{-1}\right)$, the streamwise derivative of the disturbance amplitude $\phi(x)$ imparts a parabolic character to disturbance propagation (BHS interpret the presence of this term as the defining feature of nonparallel stability theories). They therefore call their equation the Parabolic Stability Equation (PSE as it is popularly known) and solve it by marching downstream. It is relevant to point out here that in their respective formulations, both BHS and GN95 make the assumption that, since the boundary layer growth rate is $O\left(R^{-1}\right)$, downstream variations in $\phi$ are at most of this order, i.e.,
the second derivative \( \frac{d^2f}{dx^2} \), being of higher order and therefore negligible. The parabolic effects are therefore presumed to be "weak." The question we pose in this paper is, how important really are the parabolic effects, i.e., given Eq. (2), do they affect stability at a lower order than is apparent at first? If this is indeed the case, is it possible to isolate their "largest" contribution? We proceed to do so by formulating a "lowest-order parabolic" (LOP) stability equation. This equation contains all effects up to and including \( O(R^{-2}) \). It is shown that stability results obtained from the LOP equation are close to those from a more extended nonparallel analysis of the PSE type.

The present approach places great stress on achieving a rational approximation, i.e., all effects up to a given order in the reciprocal of the Reynolds number are included, and higher-order effects are neglected. Second, compared to BHS, the streamwise derivative is interpreted somewhat differently, leading to different solution methods. The present lowest-order parabolic equation is a subset of the "full" nonparallel equation of GN95, which includes all terms of \( O(R^{-1}) \) in an appropriate primitive equation for the disturbance stream function; since the equation of BHS also includes all terms of this order in addition to others, a comparison between these two equations (GN95 and BHS) brings out the differences in the approaches. The two equations are written out in full and compared at length in Sec. II; for clarity of discussion we introduce here their structure alone. The full nonparallel equation of GN95 has the form

\[
\{\text{OS}\} \phi = \frac{1}{R} \{NP_1 + N_2\} \phi = 0 \left( \frac{1}{R} \right)^2.
\]  

(3)

with the Orr–Sommerfeld operator \( \{\text{OS}\} \) containing certain terms of \( O(1) \) and others with a factor \( R^{-1} \). The operator \( \{NP_1\} \) consists of nonparallel terms due to the change in the boundary-layer thickness, streamwise variations in the free-stream velocity as well as the \( x \)-dependence of the disturbance. The operator \( \{NP_2\} \) accounts for higher-order corrections to the mean flow, due to displacement thickness, surface curvature, etc. (the effect of displacement thickness on the mean flow for Falkner–Skan wedge flows was considered by GN95). Equation (3) includes all terms with the factor \( R^{-1} \) in the primitive variables, and will be termed "nominally" correct to \( O(R^{-1}) \) in the following. The PSE, on the other hand, may be written in the present notation as

\[
\{\text{OS}\} \phi + \frac{1}{R} \{NP_1\} \phi + \frac{1}{R} \{NP_2\} \phi = O \left( \frac{1}{R} \right)^2.
\]  

(4)

contains certain terms of a higher order \( (R^{-2}) \) but omits others of the same order when written out in the variables of GN95. A more consistent version of the PSE has been derived by Simen et al., in which the higher-order terms are dropped. Since BHS derive their equation for a semi-infinite flat plate, for which \( O(R^{-1}) \) mean flow effects happen to vanish, no operator like \( \{NP_1\} \) appears in their equation, but of course in more general flows the operator \( \{NP_2\} \) cannot be ignored.

A major qualitative difference between these equations and the Orr–Sommerfeld equation,

\[
\{\text{OS}\} \phi = 0,
\]  

(5)

is that while Eq. (5) is an ordinary differential equation in \( y \), the "nonparallel" equations (3) and (4) contain derivatives with respect to \( x \) as well. The PSE, as mentioned before, is solved by space marching. GN95, on the other hand, noting that \( \partial \phi / \partial x \) is independent of the streamwise coordinate \( x \) to the order considered, treat it as a perturbation on an ordinary differential equation, and solve Eq. (3) by a trial and error procedure. In the case of a boundary layer over a semi-infinite flat plate, the two methods when applied to the same equation lead to virtually identical solutions; details of the differences in the equations and approaches are discussed in the following sections.

The LOP theory assumes importance in the light of the following discussion. The mean flow contains contributions of \( O(R^{-1}) \) except in the special case of flow over an ideal semi-infinite flat plate. A stability analysis conducted using a full nonparallel equation including all terms of \( O(R^{-1}) \) would be rational only if the mean flow were correct up to this order. Apart from it being not always feasible for the mean flow to be prescribed to this degree of accuracy, it is obvious that nonparallel effects must exist even when only the lowest-order contributions to the mean flow are considered. However, the OS does not arise as the correct equation in any rational approximation of the full boundary-layer stability equation: the equation correct to the lowest order is that formulated in GN79. A rational stability theory (LOP) correct to the next higher order is formulated in this paper and it is shown that the LOP is the highest-order theory consistent with a low-order mean flow. Since most stability analyses are conducted using low-order mean flow, the LOP, being considerably simpler than the PSE, is of practical utility.

The suggestion implicit in the above statements is that the spatial development of the flow affects stability at a lower order than \( O(R^{-1}) \). A rational stability theory considering all effects including \( R^{-2} \) was formulated by GN79; with neutral stability defined correctly for the flow quantity being studied, this theory was shown to contain most nonparallel effects. Up to this order, there is no explicit effect of the downstream propagation of the disturbances on the stability. Indeed, a legitimate question about a theory of this type is the following: if an ordinary differential equation in \( y \) [like Eq. (5)] has a solution \( \phi(y) \), an arbitrary function of \( x \) times \( \phi(y) \) is also a solution; so how does the \( x \)-dependence get determined? In practice this question has been answered, e.g., in \( e^x \)-type calculations, by noting that an ODE like the Orr–Sommerfeld equation, through the dependence of \( R \) on \( x \) carries \( x \) as a parameter. Thus the amplitude of the disturbance at any station is determined by the amplification or attenuation that it suffers through the stability characteristics (computed assuming parallel flow) at the immediately preceding station. A more satisfactory answer to this question
appears in their equation, but the operator \([NP]\) contains no dependence between these equations.

The basic equation in \(V_{t}\) and \(V_{r}\) contains derivative terms, \(y_{t}\) and \(y_{r}\), respectively, are

\[
\frac{dp}{d\theta} = 0 \quad \text{for} \quad y = 0
\]

and

\[
y = 0, \quad D_{r}y = 0 \quad \text{as} \quad y \rightarrow \infty
\]

in which the operators \([OS]\), \([NP]\) and \([NP_{h}]\), respectively, are

\[
\{OS\} = \left\{ (\omega - \alpha \phi_{t}^{h})(D^{2} - \alpha^{2}) + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4}) \right\}
\]

\[
\{NP\} = \left\{ \rho \phi_{r}D^{2} + (2\eta - \rho)\phi_{r}D^{2} + [2y_{t}\phi_{r} + (2\eta - \rho)\phi_{r}D^{2} + [\omega - \alpha \phi_{t}^{h} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4})] \phi_{r} - \frac{3}{2}(\rho - \phi)(\alpha \phi_{r}^{h})^{2} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4}) \phi_{r}^{h} \right\}
\]

\[
\{NP_{h}\} = \left\{ \rho \phi_{r}D^{2} + (2\eta - \rho)\phi_{r}D^{2} + [2y_{t}\phi_{r} + (2\eta - \rho)\phi_{r}D^{2} + [\omega - \alpha \phi_{t}^{h} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4})] \phi_{r} - \frac{3}{2}(\rho - \phi)(\alpha \phi_{r}^{h})^{2} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4}) \phi_{r}^{h} \right\}.
\]

II. THE BASIC EQUATIONS

The full nonparallel equation of GN95 is given by Eq. (3) with the boundary conditions

\[
\phi = D_{r}\phi = 0 \quad \text{at} \quad y = 0
\]

and

\[
y = 0, \quad D_{r}y = 0 \quad \text{as} \quad y \rightarrow \infty
\]

in which the operators \([OS]\), \([NP]\) and \([NP_{h}]\), respectively, are

\[
\{OS\} = \left\{ (\omega - \alpha \phi_{t}^{h})(D^{2} - \alpha^{2}) + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4}) \right\}
\]

\[
\{NP\} = \left\{ \rho \phi_{r}D^{2} + (2\eta - \rho)\phi_{r}D^{2} + [2y_{t}\phi_{r} + (2\eta - \rho)\phi_{r}D^{2} + [\omega - \alpha \phi_{t}^{h} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4})] \phi_{r} - \frac{3}{2}(\rho - \phi)(\alpha \phi_{r}^{h})^{2} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4}) \phi_{r}^{h} \right\}
\]

\[
\{NP_{h}\} = \left\{ \rho \phi_{r}D^{2} + (2\eta - \rho)\phi_{r}D^{2} + [2y_{t}\phi_{r} + (2\eta - \rho)\phi_{r}D^{2} + [\omega - \alpha \phi_{t}^{h} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4})] \phi_{r} - \frac{3}{2}(\rho - \phi)(\alpha \phi_{r}^{h})^{2} + \frac{1}{R}(D^{4} - 2\alpha^{2}D^{2} + \alpha^{4}) \phi_{r}^{h} \right\}.
\]
which contains all effects up to and including $O(R^{-m})$:

any loss of accuracy. It may further be noted that the solutions of GN95 agree well with those of BHS, which shows that the higher-order terms in Eq. (17) separate out because of the use of similarity variables in GN95. The approach of GN95 has the advantage that it illuminates connections with Orr-Sommerfeld and similar ordinary differential equations.

III. A HIERARCHY OF STABILITY EQUATIONS

The objective of this section is to derive a sequence of composite stability equations of order lower than $R^{-1}$, each composite equation being capable of providing uniformly valid solutions to an appropriate order throughout the boundary layer. We begin by rewriting the full nonparallel equation (3) in the following form:

$$\begin{align*}
\{NP\} &= 2i\alpha R(D^2 - \alpha^2) + 2\alpha^2 D - 2\gamma_2 D - \alpha^2 \\
&= 4i\alpha R(D^2 - \alpha^2) + \frac{\partial^2}{\partial x^2} 2i\alpha R(D^2 - \alpha^2). (17)
\end{align*}$$

Since Eq. (4) is written for a Blasius mean flow, the operator $\{NP\}$ must be simplified using the fact that $p = q$ in this case for a proper comparison. When this is done it will be seen that Eq. (4) includes all terms of $O(R^{\infty})$ in Eq. (3), but contains in addition several higher-order terms. The primitive equations underlying the full nonparallel theory of GN95, however, are the same as the PSE; the higher-order terms in Eq. (17) separate out because of the use of similarity variables in GN95. The approach of GN95 has the advantage that it illuminates connections with Orr-Sommerfeld and similar ordinary differential equations. It may further be noted that the solutions of GN95 agree well with those of BHS, which shows that the higher-order terms in Eq. (17) are unnecessary, and may be omitted, as expected, without any loss of accuracy.

GN97 formulated the lowest-order stability equation which contains all effects up to and including $O(R^{-12})$:

$$\begin{align*}
\{NP\} &= 2i\alpha R(D^2 - \alpha^2) + 2\alpha^2 D - 2\gamma_2 D - \alpha^2 \\
&= 4i\alpha R(D^2 - \alpha^2) + \frac{\partial^2}{\partial x^2} 2i\alpha R(D^2 - \alpha^2). (17)
\end{align*}$$

This equation has been derived by considering in turn the critical layer, the wall layer, and the rest of the boundary layer, ordering terms respectively in each of these, and by constructing a minimal composite equation containing all the terms which are $O(R^{-12})$ or lower in any of the layers: the details are given in the following section. Equation (18) has the form of a modified OS equation. The differences between Eq. (18) and the OS equation (5) have been discussed in GN97, where it is shown that stability results from Eq. (18) are critical to the stability of nonparallel flow in the boundary layer above the critical layer as well as in between the critical and wall layers when these are distinct. In accordance with Eq. (20), the eigenfunction is expressed as an asymptotic expansion in each layer, respectively, as

$$\begin{align*}
\phi(y) &= \sum_{k=1}^{\infty} \sum_{m=1}^{\infty} e^{s_k} \phi_k(y), & k = 0, 1, 2, \ldots, m = 1, 2, \ldots, (21)
\end{align*}$$

with

$$\begin{align*}
\eta = \eta_c + \gamma y_c, & \quad \eta_c = \left(\frac{\eta}{\varepsilon_c}\right), & \quad \gamma = \frac{\varepsilon_c}{\eta_c}, (24)
\end{align*}$$

$y_c$ is the critical height, at which the mean velocity (including higher-order contributions) equals the phase velocity of the disturbance:

$$\begin{align*}
c &= \frac{\alpha}{\gamma} = \Phi_c. (25)
\end{align*}$$
The mean streamfunction $\Phi$ has already been written as the asymptotic expansion (13). The higher-order component of the mean flow in this equation, $\Phi_1$, contains contributions due to the displacement effects, surface curvature, etc. (see, e.g., Van Dyke). In the flow over a semi-infinite wedge, $\Phi_0$ satisfies the Falkner-Skan equation, while $\Phi_1$ arises due to the displacement alone, and satisfies Eq. (17) of GN95. The lower-order component $\Phi_0$ can be expanded in the critical and wall layers, respectively, as

$$\Phi_0 = \Phi_{0w} + \frac{(y-y_c)^2}{2} \Phi_{0w}^{(2)} + \cdots$$

and

$$\Phi_{0w} = \frac{y^2}{2} \Phi_{0w}^{(2)} + \cdots.$$  \hspace{1cm} (27)

Substituting Eqs. (21) and (13) into the full nonparallel Eq. (19), and equating terms of like order in $\epsilon$, we get the lowest-order equation for the outer layer, which is simply the Rayleigh equation

$$\left( \omega - \alpha \Phi_0'' \right) \chi_0 - \alpha' \chi_0' + \alpha \Phi_0' \chi_0 = 0.$$  \hspace{1cm} (28)

From the corresponding equation of the higher order we get

$$\epsilon = (\alpha R)^{-1}.$$  \hspace{1cm} (29)

In the critical layer, the lowest-order equation

$$\chi_0'' + i \eta_0 \Phi_0' \chi_0 = 0.$$  \hspace{1cm} (30)

is obtained by setting

$$\epsilon = (\alpha R)^{-1}.$$  \hspace{1cm} (31)

In Eq. (30), only the lower-order component of the mean flow appears. This is because, using Eq. (25), the factor $e^{-\Phi_0'}$ in Eq. (19) can be expanded in the critical layer as

$$e^{-\Phi_0'} = (y-y_c)^2 \Phi_0' \frac{y-y_c}{2} \Phi_0' + \cdots.$$  \hspace{1cm} (32)

Using Eqs. (24) and (13), and arranging in increasing powers of $\epsilon$, it is noticed that terms containing derivatives of the higher-order mean flow, $\Phi_1$, appear at higher orders in the expansion, and $\Phi_0''$ and $\Phi_0'''$ in the first two terms on the right-hand side of Eq. (32) may be replaced, respectively, by $\Phi_0''$ and $\Phi_0'''$, up to the order considered.

In the wall layer, with

$$\chi_0 = e^{-i\alpha R} \chi_0$$  \hspace{1cm} (33)

we get

$$\chi_0'' + i \eta_0 \Phi_0' \chi_0 = 0.$$  \hspace{1cm} (34)

On examining Eq. (20) in the light of Eqs. (29), (31), and (33), it is seen that an appropriate choice of $\epsilon$ is $(\alpha R)^{-1}$. In Eqs. (21) and (23), the $X$ functions can be set to zero without any loss of generality, and Eq. (20) may be written as

$$\phi = x_0 + \epsilon x_1 + \epsilon^2 \log(\epsilon) x_1 + \epsilon x_2 + \epsilon^2 x_2 + O(R^{-1}).$$  \hspace{1cm} (35)

The subsequent terms in Eq. (35) can be derived from Eq. (19) by the same procedure, and are solutions of the following equations:

$$X_{11}^{(1)} - i \eta_0 \Phi_0' X_{11} + i \eta_0 \Phi_0' X_{11} - \rho \Phi_0 X_{11} = 0,$$  \hspace{1cm} (36)

$$X_{12}^{(1)} - i \eta_0 \Phi_0' X_{12} + i \eta_0 \Phi_0' X_{12} - \rho \Phi_0 X_{12} = 0,$$  \hspace{1cm} (37)

$$X_{11}^{(2)} - i \eta_0 \Phi_0' X_{11} + i \eta_0 \Phi_0' X_{11} - \rho \Phi_0 X_{11} - i \eta_0 \Phi_0' X_{11} = 0,$$  \hspace{1cm} (38)

$$X_{12}^{(2)} - i \eta_0 \Phi_0' X_{12} + i \eta_0 \Phi_0' X_{12} - \rho \Phi_0 X_{12} = 0.$$  \hspace{1cm} (39)

For reasons discussed earlier, the higher-order mean flow does not appear in any of the equations derived here.

From an examination of the above set of equations, it is evident [from the presence of the term containing the factor $p$ in Eq. (36)] that the lowest-order nonparallel effect is already present in the equation for $X_{11}$ which has an $O(R^{-1})$ contribution. It is also apparent, from Eq. (39), that the streamwise derivative of the disturbance eigenfunction first appears in the coefficient of $\epsilon^2$, i.e., at $O(R^{-2})$.

We now return to the full nonparallel Eq. (3) and construct from it a "minimal" subset containing just those terms necessary for deriving equations for $X_0$ through $X_2$ and $\lambda_1$ and $\lambda_2$ in Eq. (20). This works out to be the lowest-order stability Eq. (18). Incidentally, the terms contained in Eq. (18) already include the equation for $X_0$ which makes the lowest-order stability equation correct to $O(R^{-1})$. The next composite stability equation in this hierarchy would include $X_4$ and $\lambda_3$ and is given by

$$\left( \omega - \alpha \Phi_0'' \right) D^2 + \alpha \Phi_0' + \frac{i}{R} \left( D^2 + p \Phi_0 D \right)$$  \hspace{1cm} (40)

which includes the streamwise derivative of the disturbance amplitude, which was absent in the lowest-order Eq. (18), i.e., the effects of the parabolic nature of the flow on its stability first appear in this...
equation. It is therefore appropriate to call it the "Lowest-order Parabolic Stability Equation" (LOP equation). The boundary conditions are given by Eqs. (6) and (7).

It is important to note that the highest-order contributions to the mean flow, i.e., $\Phi_i$, and so on, do not affect stability up to the order considered. From Eq. (35) it is clear that the next stability equation in the hierarchy would be one correct to $O(R^{-1})$, such as the full nonparallel equation of GN95. If we were to use such an equation, in order to be consistent we need to know $\Phi_i$ accurately. The LOP is thus the highest-order stability equation that is consistent with the lowest-order approximation to the mean flow in the boundary layer.

When the lowest-order parabolic equation is compared to the OS Eq. (5), it is noticed that the term $\alpha^\prime \Phi_i$, which is present in the Orr–Sommerfeld equation, being $O(R^{-1})$ or higher everywhere in the boundary layer, has to be neglected in this analysis. Instead, the term containing $D_3^2\Phi$ and two additional terms containing $D_1^2\Phi$ are now included. As discussed in GN97, the third derivative term is due to the advection of the disturbance vorticity, $\zeta$, by the normal component of the mean velocity. The nonparallel component of the streamwise advection of $\zeta$, on the other hand, gives rise to a new second derivative term as well as to the explicit parabolic term in Eq. (41). Equation (41) is a low-order subset of Eq. (3), which arises out of the vorticity equation for linear disturbances,

$$\frac{D\Phi_d}{Dt} = \frac{\partial \Phi_d}{\partial \xi} + \frac{\partial \Phi_d}{\partial \eta} + \frac{\partial \Phi_d}{\partial t} = 0,$$

(42)

Here $D/Dt$ stands for the total derivative following the mean flow. The origin of each of the terms in Eq. (41) can be traced back to a corresponding term in Eq. (42) and the primitive equation for Eq. (41) may be derived to be

$$\frac{D\Phi_d}{Dt} = \frac{\partial \Phi_d}{\partial \xi} + \frac{\partial \Phi_d}{\partial \eta} + \frac{\partial \Phi_d}{\partial t} = \frac{\partial^2 \Phi_d}{\partial x^2} + \frac{\partial^2 \Phi_d}{\partial y^2} + \frac{\partial^2 \Phi_d}{\partial t^2}.$$

(43)

Equation (43) contains all nonparallel effects up to $O(R^{\prime 2})$; it is therefore sufficient to begin from Eq. (43) instead of (42) in order to obtain stability characteristics up to this order. This observation is relevant especially for nonsimilar flows where Eq. (41) will not hold. In comparison with the primitive equation for the lowest-order stability equation [Eq. (4.5) of GN97], it is seen that the only additional term in Eq. (43) is the last one, which represents streamwise diffusion of the dominant term in disturbance vorticity. In the LOP equation (as in the OS equation), this diffusion appears as an additional second derivative term ($-2\sigma^\prime D_1^2\Phi$). Note that the last term in Eq. (43) is significant only at the critical layer, where the dominant contribution to $\zeta_d$ comes only from $\partial \Phi_d/\partial \eta$; the other term $\partial \Phi_d/\partial x$ will be of higher order.

IV. RESULTS

Neutral stability boundaries were computed using both the full nonparallel Eq. (3) and the LOP Eq. (41) for various heights in the boundary layer using the finite difference algorithm and solution procedure described in GN95. The main objective of this comparison is to enable an assessment of how accurate the present theory is in accounting for the effects of flow nonparallelism; as already pointed out, the "full nonparallel" theory [Eq. (6) of GN95] is to be considered consistent in any flow problem only if higher-order effects in boundary layer mean profiles vanish. The stability boundary at the inner maximum of the streamwise disturbance velocity, as predicted by different theories for the flow over a flat plate, is plotted in Fig. 1; here $F$ is the nondimensional frequency parameter which is proportional to the dimensional frequency $\omega_d$. (For a Falkner–Skan mean flow with a pressure gradient parameter $\beta$, we have $F = \beta(1-\beta)^{-1/2} \omega_d$.) The results of BHS are practically indistinguishable from those of the full nonparallel Eq. (3) of GN95 and are therefore not indicated separately in the figure. Also not shown are the DNS results of Fusel and Kanelmann, with which the full nonparallel results are in excellent agreement.

It is well-known that in nonparallel flows, assessment of stability depends on the path along which the disturbance is monitored. At a given $\beta$, for example, the amplitude of $\Phi_i$ may decay along one path and grow along another. Fusel and Kanelmann and GN95 have shown that stability characteristics are very sensitive to the normal distance from the wall of the monitoring location; the variation of the critical Reynolds number with height above the wall, for example, is certainly much larger in magnitude numerically (i.e., is of lower order in an asymptotic expansion) than $R^{-1}$. The dependence of stability on height thus is a good check for the performance of the LOP theory. We therefore compute stability boundaries at various heights in the boundary layer, and obtain from these two characteristic quantities at each height, namely the highest possible frequency $F_{max}$ which instability is possible, and the critical Reynolds number $R_c$. The highest unstable frequency is plotted as a function of height in Fig. 2. It is evident that the LOP, in spite of being a much simpler equation than Eq. (3), is in close agreement with the predictions of the latter. As discussed in GN97, the conventional OS results shown in the figures are independent of height. It is, however, possible to obtain height dependence from the OS by taking suitable directional
lies at the inner maximum. So!:: Serveld.

e derivatives, as shown by BHS in their Fig. 5 and discussed in GN95.) The conventional OS result, which gives one answer across the boundary layer, lies very close to the result of the full nonparallel theory at the inner maximum of \( \phi_y \). In spite of not being rational up to any order, the performance of the Orr-Sommerfeld theory is quite remarkable, at least in the vicinity of the inner maximum.

The same quantities as plotted in Figs. 1 and 2 are shown in Figs. 3 and 4, respectively, for a boundary layer in a strong adverse pressure gradient. For pressure gradient flows, there is a need for results from DNS studies (or experiment) against which different stability theories can be assessed. In the absence of such data, a stability boundary correct to \( O(R^{-1}) \) for this flow, with a Falkner-Skan parameter value of \( m = -0.06 \), was computed by GN95, and is reproduced in Fig. 3. The other stability boundaries, being correct to lower orders, contain only the lowest-order mean flow corresponding to \( m = -0.06 \). The LOP is seen to perform much better than the lowest-order ODE of GN97. Shown in Fig. 5 is the variation of the critical Reynolds number with height. Figures 4 and 5 show that the LOP is able to follow closely the height dependence displayed by the full nonparallel theory. The predictions of the highest unstable frequency are excellent. Those of the critical Reynolds number, although showing slightly greater discrepancy, are still very good. In these two figures, all the results have been obtained using the lowest-order mean flow.

We have repeated these calculations for different values of \( m \). The results are summarized in Figs. 6 and 7, where the fractional error (when compared to the full nonparallel solution with lower-order mean flow), respectively, in the critical
Reynolds number and the highest unstable frequency, are plotted against the pressure gradient; even in high adverse pressure gradients, the error is $\leq 10\%$, while the error is much lower in favorable pressure gradients, presumably due to the high Reynolds numbers involved. As discussed in GN95, the streamwise derivative $\frac{\partial \phi}{\partial x}$ is arrived at by trial and error: in computations of full nonparallel solutions, numerical convergence is a problem, especially in adverse pressure gradients. As a result, some jitter was evident in the plots of neutral stability close to the critical Reynolds number. This is important and is useful in assessing different stability theories, but, as shown in GN97, the stability loops at different heights can exhibit some rather peculiar features. A related but more important point is that, in nonparallel flows, there is a fundamental problem in apportioning streamwise variation in disturbance amplitude between exponential growth [through the imaginary part of $\alpha$ in Eq. (12)] and streamwise evolution [through the dependence on $x$ of $\phi(x,y)$]. It is therefore perhaps more appropriate to examine directly the streamwise variation of the disturbance amplitude itself. Furthermore, for making predictions about the onset of transition to turbulence, it is again the amplitude of a given initial disturbance as a function of the streamwise location that is of primary relevance.

Quantitative differences in $\log[\frac{\Delta A}{A_{\text{max}}}]$ obtained from different stability theories tend to be smaller than differences in stability boundaries, even in adverse pressure gradient flows, and the resulting $\eta$ factor varies little (GN95). This leads to the conclusion that lower-order theories, especially the LOP, should be sufficient for computations of $\eta$; at any rate, the greater accuracy of higher-order theories is of significance only when higher-order mean velocity profiles are available. Amplitudes computed using the full nonparallel Eq. (3) were shown by GN95 to be in excellent agreement with the results of the direct numerical simulations of Fasel and Konzelmann. Figure 8 shows the performance of the present theory in the computation of amplification factors at the inner maximum of the eigenfunction. The maximum discrepancy when compared to the full nonparallel solution is less than 0.1, but this difference should be much smaller in the envelopes of disturbance amplitudes, i.e., in the $\eta$ factor. The LOP is in closer agreement with the full nonparallel results than the lowest-order results are. This is true even in adverse pressure gradients, the error is $\geq 10\%$, while the error is much lower in favorable pressure gradients.

In the results presented so far, the emphasis has been on the neutral stability boundary. This is important and is useful in assessing different stability theories, but, as shown in GN97, the stability loops at different heights can exhibit some rather peculiar features. A related but more important point is that, in nonparallel flows, there is a fundamental problem in apportioning streamwise variation in disturbance amplitude between exponential growth [through the imaginary part of $\alpha$ in Eq. (12)] and streamwise evolution [through the dependence on $x$ of $\phi(x,y)$]. It is therefore perhaps more appropriate to examine directly the streamwise variation of the disturbance amplitude itself. Furthermore, for making predictions about the onset of transition to turbulence, it is again the amplitude of a given initial disturbance as a function of the streamwise location that is of primary relevance.

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In steady parallel flows, there is a finite streamwise variation in the potential distribution [through the solution of the potential equation \( \phi(x,Y) \)], which is therefore affected directly by the streamwise variation in the amplitude of oscillations. It is therefore possible to get the full benefit of higher-order effects, except for the solution of the displacement thickness, which is the same as that of the full nonparallel system of \( F = 5 \). For compressible mean flow, due for example to displacement thickness or pressure gradients, as shown in Fig. 9 for the Falkner-Skan flow at \( \theta = 0 \), again the difference in amplification factor is less than 0.1.

V. CONCLUSIONS

We have derived and analyzed here the highest-order stability equation lower than the full nonparallel system of \( F(\theta) \), which includes both parallel and nonparallel terms. The solution at large \( \theta \) is therefore the same as that of the full nonparallel system. In excellent agreement with previous calculations of Drazin and Reid6 and its full nonparallel solution in the critical layer, which matches with the second term in Eq. (A1) and

\[
[x_{0B}] = \left[ \frac{\Phi_{m}''}{2\Phi_{m}} \right]_{0} \eta \log \eta.
\]

which, together with

\[
[x_{0C}] = \left[ \frac{\Phi_{m}''}{2\Phi_{m}} \right]_{0} \eta.
\]
matches the term containing \((\gamma - \gamma_c) \log(\gamma - \gamma_c)\) in Eq. (A.2). By the same argument, nonparallel terms in Eq. (39) are not important at large \(\eta_c\), and the matching obeys classical theory. The two solutions relevant for matching with the outer layer are

\[
[\lambda_2](\eta_c \rightarrow \infty) \sim \left[ \frac{\Phi_{nc}}{\Phi_{nc} + \alpha^2} \right] \eta_c^3
\]

(A7)

and

\[
[\lambda_2](\eta_c \rightarrow \infty) = \left[ \frac{\Phi_{nc}}{2\Phi_{nc} + \alpha^2 - \left( \frac{\Phi_{nc}}{\Phi_{nc}} \right)^2} \right] \eta_c^2
\]

(A8)

Equation (40) for \(\lambda_{1,2}\) is the same as Eq. (36), whose solution has the same form as Eq. (A4):

\[
[\lambda_{1,2}](\eta_c \rightarrow \infty) = \frac{\Phi_{nc}}{2\Phi_{nc}} \eta_c^2.
\]

(A9)

It can easily be verified that these solutions match with the third terms of Eqs. (A1) and (A2) respectively. For the wall layer, all solutions of Eqs. (34) and (38) decay outside the wall layer and do not appear in the matching process.