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Inelastic Post-Buckling of Columns

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Introduction

Slender cantilever columns find extensive use as struts carrying compressive loads. In the optimum design of these members, it is necessary to take into account both geometrical and physical nonlinearities in the analysis. Oden and Childs (1970) studied the post-buckling of a nonlinearly elastic bar of uniform cross-section with a compressive tip load by assuming a moment-curvature relationship which simulated that of a class of elasto-plastic material. Verma and Krishnamurty (1974) studied the post-buckling problem for tapered beams of rectangular cross section incorporating material nonlinearity in terms of Ramberg-Osgood stress-strain relations (Ramberg and Osgood, 1943) using a finite-difference approach with reduced modulus concept. Recently, Monasa (1974, 1977) studied the large deformation of very slender elasto-plastic bars using a nonlinear moment-curvature relationship (Beedle, 1966) to represent the behavior in the plastic zone. The important assumption in the investigation by Monasa (1974, 1977) is that the axial stress is negligible compared to bending stress and may be neglected.

The concept of integrating through the thickness to account for the stress reversal has been extensively applied before and it is the common practice. In this Note, the earlier procedure of integrating through the thickness as indicated by the present authors (1976), is used to determine the axial load and bending moment that each section can carry, and this is further used to compute the state of deformation in the structure. This indirect numerical iterative analysis allows the post-buckling behavior of a structure for nonlinearly elastic material to be

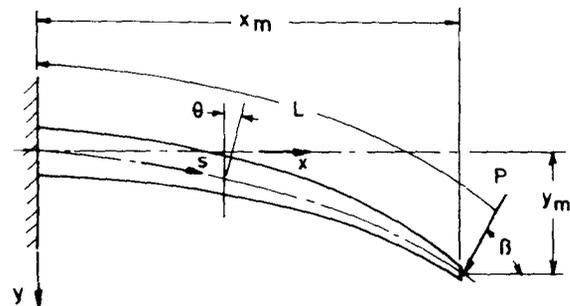


Fig. 1 Cantilever column: coordinate system

determined as function of the parameters in the stress-strain laws.

Analysis

The deformed configuration and the coordinate system used here for the cantilever column with an inclined tip load is shown in Fig. 1. A class of columns of variable rectangular cross-section, of depth $D(s)$ and breadth $B(s)$ are chosen in the form $D(s) = D_o d(s)$ and $B(s) = B_o b(s)$. The Ramberg-Osgood relation (Ramberg and Osgood, 1943) is rewritten in an alternate form similar to that suggested by Rao and Krishnamurty (1971) as

$$(\sigma/\sigma_r) = f(\epsilon) = [(\epsilon/\epsilon_r) - 0.4(\epsilon/\epsilon_r)^m] \quad (1)$$

where σ_r , ϵ_r , m are parameters that determine the nonlinearity, and σ , ϵ are the stress and strain, respectively. In a deflected stable equilibrium position, the total compressive stress is

$$\sigma_o(\eta) = P \cos(\beta + \theta) / (B_o D_o b d) \quad (2)$$

and bending moment is

$$M(\eta) = P[(x_m - x) \sin \beta + (y_m - y) \cos \beta].$$

Normal force and bending moment resisted by the section are

$$B_o D_o b d \sigma_r \int_{-\alpha_1}^{\alpha_2} f(\epsilon_o + \lambda \alpha) d\alpha \text{ and}$$

$$B_o D_o^2 b d^2 \sigma_r \int_{-\alpha_1}^{\alpha_2} f(\epsilon_o + \lambda \alpha) \alpha d\alpha, \text{ respectively.}$$

Here $\lambda = (D/L)(d\theta/d\eta)$, $\eta = s/L$, $\alpha = (z/D)$, where z is the distance measured from neutral axis. We know from equa-

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$$\frac{\sigma_o(\eta)}{\sigma_r} = f(\epsilon_o) = \frac{P}{\sigma_r B_o D_o} \frac{\cos(\beta + \theta)}{bd} \quad (3)$$

From equilibrium consideration, we have from above equations

$$\int_{-\alpha_1}^{\alpha_2} f(\epsilon_o + \lambda\alpha) d\alpha = \frac{P}{\sigma_r B_o D_o} \frac{\cos(\beta + \theta)}{bd} \quad (4)$$

and

$$\int_{-\alpha_1}^{\alpha_2} f(\epsilon_o + \lambda\alpha) \alpha d\alpha = \frac{PL}{\sigma_r B_o D_o^2} \left[\left(\frac{x_m}{L} - \frac{x}{L} \right) \sin\beta + \left(\frac{y_m}{L} - \frac{y}{L} \right) \cos\beta \right] \quad (5)$$

Following the procedure given by Verma and Krishnamurty (1974) and defining $f(\epsilon) = (\epsilon/\epsilon_r) - 0.4(\epsilon/\epsilon_r)^m$ in the loading or concave side of the beam, and $f(\epsilon) = (\epsilon/\epsilon_r) - 0.4(\epsilon_o/\epsilon_r)^m$ in the unloading or convex side, one obtains after some algebra, the equation

$$\int_{-\alpha_1}^{\alpha_2} \left[\left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right) - 0.4 \left(\frac{\epsilon_o}{\epsilon_r} \right)^m \right] d\alpha + \int_{\alpha_2}^{\alpha_1} \left[\left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right) - 0.4 \left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right)^m \right] d\alpha = \frac{P}{\sigma_r B_o D_o} \frac{\cos(\beta + \theta)}{bd} \quad (6)$$

Equation (3) may be rewritten as

$$\int_{-\alpha_1}^{\alpha_2} f(\epsilon_o) d\alpha = \int_{-\alpha_1}^{\alpha_2} \left[\frac{\epsilon_o}{\epsilon_r} - 0.4 \left(\frac{\epsilon_o}{\epsilon_r} \right)^m \right] d\alpha = \frac{P}{\sigma_r B_o D_o} \frac{\cos(\beta + \theta)}{bd} \quad (7)$$

From equations (6) and (7), we obtain

$$\frac{\lambda}{\epsilon_r} \int_{-\alpha_1}^{\alpha_2} \alpha d\alpha + \int_{\alpha_2}^{\alpha_1} 0.4 \left[\left(\frac{\epsilon_o}{\epsilon_r} \right)^m - \left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right)^m \right] d\alpha = 0$$

Noting that $\alpha_1 + \alpha_2 = I$, the above equation can be rewritten as

$$\frac{\lambda}{2} (I - 2\alpha_2) = \int_{\alpha_2}^{\alpha_1} 0.4 \left[\left(\frac{\epsilon_o}{\epsilon_r} \right)^m - \left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right)^m \right] d\alpha \quad (8)$$

Equation (8) determines the location of the neutral axis. For the linear case, the term on the right hand side vanishes, so that $\alpha_2 = 0.5$. We therefore get an implicit nonlinear equation in α_2 as

$$\alpha_2 = 0.5 - \frac{0.4\epsilon_r}{\lambda} \left\{ \left(\frac{\epsilon_o}{\epsilon_r} \right)^m \alpha_2 - \frac{\epsilon_r}{X(m+1)} \left[\left(\frac{\epsilon_o + \lambda\alpha_2}{\epsilon_r} \right)^{m+1} - \left(\frac{\epsilon_o}{\epsilon_r} \right)^{m+1} \right] \right\} \quad (9)$$

The moment-curvature relationship is now established from equation (5) as

$$\int_{-\alpha_1}^{\alpha_2} \left[\left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right) - 0.4 \left(\frac{\epsilon_o}{\epsilon_r} \right)^m \right] \alpha d\alpha + \int_{\alpha_2}^{\alpha_1} \left[\left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right) - 0.4 \left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right)^m \right] \alpha d\alpha = \frac{PL}{\sigma_r B_o D_o^2} \left[\left(\frac{x_m}{L} - \frac{x}{L} \right) \sin\beta + \left(\frac{y_m}{L} - \frac{y}{L} \right) \cos\beta \right] \quad (10)$$

which gives

$$\int_{-\alpha_1}^{\alpha_2} \left[\frac{\epsilon_o}{\epsilon_r} - 0.4 \left(\frac{\epsilon_o}{\epsilon_r} \right)^m \right] \alpha d\alpha + \frac{\lambda}{3\epsilon_r} (\alpha_2^3 + \alpha_1^3) + \int_{\alpha_2}^{\alpha_1} \left[0.4 \left(\frac{\epsilon_o}{\epsilon_r} \right)^m - 0.4 \left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right)^m \right] \alpha d\alpha = \frac{PL}{\sigma_r B_o D_o^2} \left[\left(\frac{x_m}{L} - \frac{x}{L} \right) \sin\beta + \left(\frac{y_m}{L} - \frac{y}{L} \right) \cos\beta \right] \quad (11)$$

Substituting the result from equation (7) in equation (11) again, we get

$$\frac{P}{\sigma_r B_o D_o} \frac{\cos(\beta + \theta)}{bd} \frac{(\alpha_2^3 - \alpha_1^3)}{2} + \frac{\lambda}{3\epsilon_r} (\alpha_2^3 + \alpha_1^3) + \int_{\alpha_2}^{\alpha_1} \left[0.4 \left(\frac{\epsilon_o}{\epsilon_r} \right)^m - 0.4 \left(\frac{\epsilon_o + \lambda\alpha}{\epsilon_r} \right)^m \right] \alpha d\alpha = \frac{PL}{\sigma_r B_o D_o^2} \left[\left(\frac{x_m}{L} - \frac{x}{L} \right) \sin\beta + \left(\frac{y_m}{L} - \frac{y}{L} \right) \cos\beta \right] \quad (12)$$

Thus equations (9) and (12) together with the relation $\alpha_1 + \alpha_2 = 1.0$ completely define the problem.

Solution

The problem on hand is stated as follows. Give the tip load in a nondimensional form, say $(PL/\sigma_r B_o D_o^2)$ and the geometry of the beam $b(\eta)$, $d(\eta)$, D_o/L , and the material properties ϵ_r, m and σ_r , it is required to determine the shape of the deformed configuration. Indirect numerical iterative scheme seems to be a simple, effective method of solution and is used here as described below.

The deformed configuration is prescribed initially by considering a certain value for the end slope θ_1 . The slope $\theta(\eta)$ is assumed as $\theta_1 \sin(\pi\eta/2)$, satisfying the conditions $\theta(0) = 0$ and $\theta(1) = \theta_1$. Thus, initially the value of $d\theta(\eta)/d\eta$ is equal to $(\pi\theta_1/2) \cos(\pi\eta/2)$, $\theta(\eta)$, and hence $x(\eta)$, $y(\eta)$ given by the expressions

$$x(\eta) = \int_0^\eta \cos\theta d\eta \text{ and } y(\eta) = \int_0^\eta \sin\theta d\eta$$

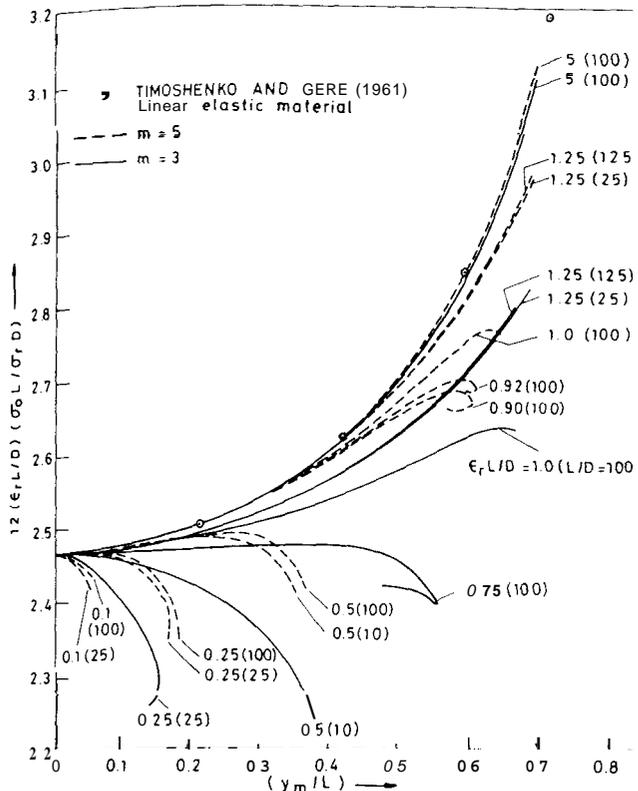
are calculated by numerical integration. It is known that $\lambda(\eta)/\epsilon_r$ is equal to $(D_o d/L\epsilon_r) (d\theta/d\eta)$ and so $\lambda(\eta)/\epsilon_r$ can be determined for the initial assumed value of $d\theta/d\eta$. The linear stress-strain approximation to equation (12), at the fixed end, gives

$$\frac{\lambda(0)}{12\epsilon_r} = \left(\frac{PL}{\sigma_r B_o D_o^2} \right) \left[\frac{\frac{x_m}{L} \sin\beta + \frac{y_m}{L} \cos\beta}{bd^2} \right]$$

As $\lambda(0)/\epsilon_r$ is known, the above equation allows to determine the value of $(PL/\sigma_r B_o D_o^2)$. From this value of $(PL/\sigma_r B_o D_o^2)$, $(P/\sigma_r B_o D_o)$ is determined and then used in equation (3) to evaluate $\sigma_o(\eta)/\sigma_r$ and hence ϵ_o/ϵ_r at each cross-section from equation (1). With these values of ϵ_o/ϵ_r and $\lambda(\eta)/\epsilon_r$, $\alpha_2(\eta)$ is calculated at each station by iteration from equation (9). Substituting the above values in equation (12) and then carrying out the integration through thickness, first as improved value of $PL/\sigma_r B_o D_o^2$ is obtained considering $\eta = 0$, using

Table 1 Numerical values at the commencement of instability, ($L/D = 100$)

m	$(\epsilon_r L/D)$	$12(\epsilon_r L/D)(\sigma_o L/\sigma_r D)$	(y_m/L)	(x_m/L)	θ_i
2	0.49	2.118	0.330	0.932	75 deg
3	0.75	2.420	0.484	0.843	75 deg
4	0.86	2.581	0.558	0.781	75 deg
5	0.92	2.686	0.598	0.740	75 deg

**Fig. 2 Load-max. deflection relation**

$\lambda(\eta)/\epsilon_r$ is computed from equation (12). The iteration is carried till the values of $PL/\sigma_r B_o D_o^2$ and $\lambda(\eta)/\epsilon_r$ converge to the required degree of accuracy.

Results and Discussions

A computer algorithm is written to carry out the numerical integration and iteration for any tapered beam with a load of arbitrary inclination β . However, only results for a uniform beam ($b=d=1$) with an axial compressive load ($\beta=0$) are presented. The Parameters governing the nondimensional tip deflection (y_m/L) and (x_m/L) to nondimensional load $\epsilon_r(\sigma_o/\sigma_r)(L/D_o)^2$ are (L/D_o) , ϵ_r and m . (Note $(\sigma_o/\sigma_r)(L/D_o) = PL/\sigma_r B_o D_o^2$). For a linear problem, where $\sigma_r/\epsilon_r = E$, we have $PL^2/EI_o = 12\sigma_o L^2/ED_o^2 = \pi^2/4$.

The load-maximum vertical deflection is plotted in Fig. 2 in the nondimensional form. It is found that the nondimensional plots of tip deflection-load is nearly identical for larger values of $(\epsilon_r L/D) = \text{constant} (> 1.0)$ for each m . This behavior corroborates the findings of Monasa (1977) that the same family of curves can be used for different parameters (σ_r/E) and (L/D) provided the product $(\sigma_r L/ED)$ is the same. Therefore, only two nondimensional parameters, m and $(\epsilon_r L/D)$ need be used to describe the problem when $(\epsilon_r L/D) > 1.0$. However, from the numerical results obtained, it could be mentioned that there is a definite deviation in the behavior for smaller values of $(\epsilon_r L/D) = \text{constant} (< 0.5)$ as shown in Fig. 2, which necessitates the use of all the three parameters.

As one expects, within the range of values considered, the decrease in m increases the material nonlinearity effect, as does the decrease in ϵ_r and L/D . Thus, the material nonlinearity effect reduces at large slenderness ratios and the geometrical nonlinearity effect predominates. This behavior is similar to that obtained for the case of inelastic large deforma-

tion of beams under a vertical tip load (Prathap and Varadan, 1976).

Computations in each case became unstable when there was a reversal in sign of stress (The stresses are based on $\lambda\alpha$ and ϵ_o . When the maximum bending stress computed from the maximum occurring value of ha is greater than the axial stress based on ϵ_o , the reversal of sign takes place and there will be a computational difficulty) and were terminated at that stage. The present analysis considered the tip angle up to 80 deg in steps of 5 deg. The values of $(\epsilon_r L/D)$ at which instability started occurring are given in Table 1 for different values of m and for a constant value of $L/D (=100)$; and this value of $(\epsilon_r L/D)$ decreases with the decrease in m . This means that the material nonlinearity and geometrical nonlinearity play an important role in the stability behavior, the instability occurring due to the shifting of neutral axis out of the domain of the beam. From the numerical results obtained, it is noticed that increase in deflection takes place for decrease in load before the commencement of the instability of the structural member.

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Stresses Around Two Rigid Cylindrical Inclusions in an Infinite Elastic Body Under Tension

S. Itou³

Introduction

Recently, fiber reinforced plastics have been widely used in designing the various members of a machine or structure because they are of high strength and are not heavy. In such materials, the loads are transmitted from the matrix, with a low degree of stiffness, to the fibers, with a high degree of stiffness, through the interfaces.

It is very complicated to obtain the stress field around a finite length circular cylindrical inclusion in an infinite elastic body. To bypass this difficulty, Kasano et al. (1981) assumed

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